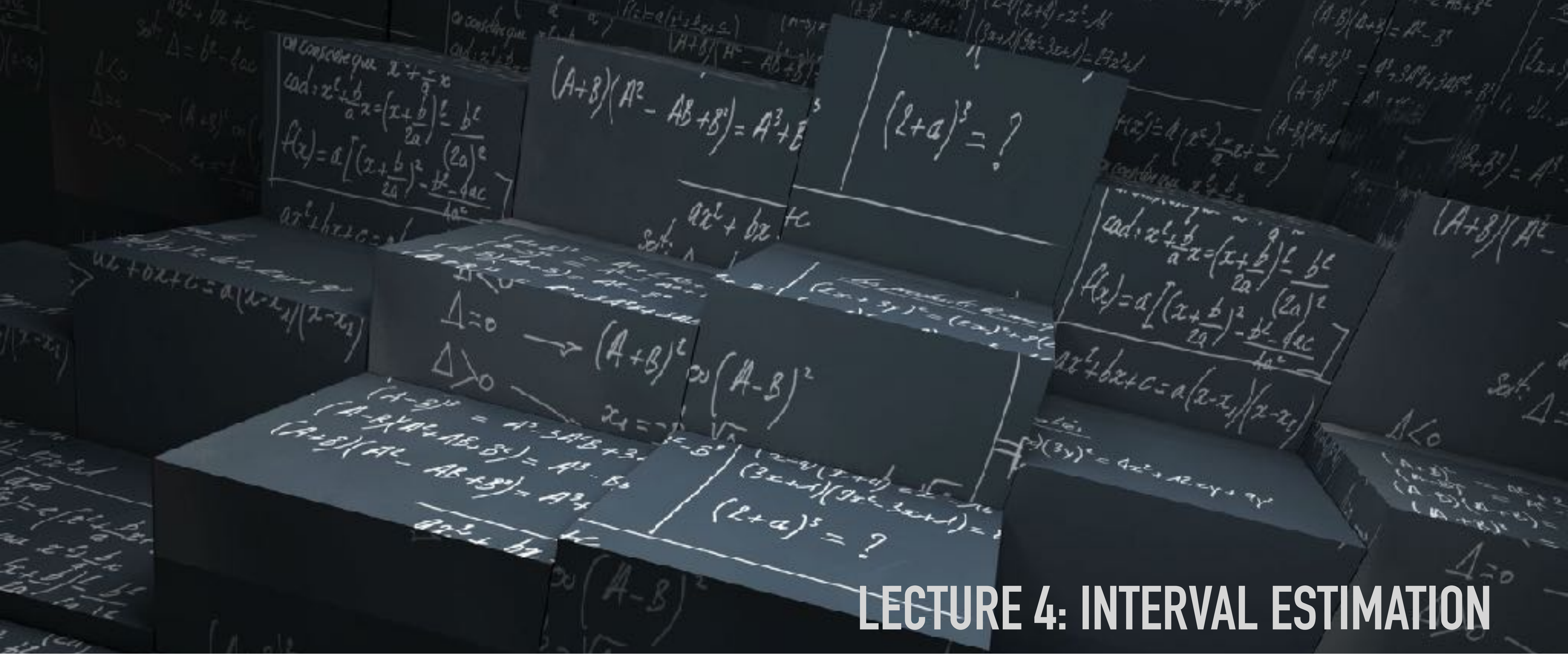

LECTURE 4: INTERVAL ESTIMATION

STATISTICAL ANALYSIS IN EXPERIMENTAL PARTICLE PHYSICS

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INTERVAL ESTIMATION

- The general goal of interval estimation –– with a given **probability β**, we want to find the range
$$\theta_a \le \theta \le \theta_b$$
which can contain the true value of the parameter of interest $\boldsymbol{\theta_0}$.
- One can choose a large probability β (e.g. 90% or 99%), such that the interval indeed contains the true value.
- One can choose β=68.3% or 95.5%, and derives the corresponding “errors” for “1σ” and “2σ”, although this is obviously only works for Normal distribution.
- Surely this estimate depends on the definitions of the probability used here. Thus the meaning of the interval, will be rather different for the **Bayesian method** and **frequentist method**.

INTERVAL ESTIMATION (CONT.)

- Different methods of interval estimation which are usually considered:
  - **Normal theory interval estimation**, a text-book/elementary frequentist method. It is an asymptotic theory, which only works when the estimates are approximately Gaussian distributed (*very often the case?*).
  - **Likelihood-based method**, which is used by Minuit and already touched during the last lecture!
  - **Neyman construction**, which is an exact frequentist method. It was developed by Jerzy Neyman et al around 1930.
  - **Bayesian interval estimation**, again, is based on the Bayes Theorem. It can be treated as a relatively straightforward(?) extension of the Bayesian point estimation. One has to take care of the prior problems as usual.

Let's start with Bayesian method this time!

INTERVAL ESTIMATION: BAYESIAN

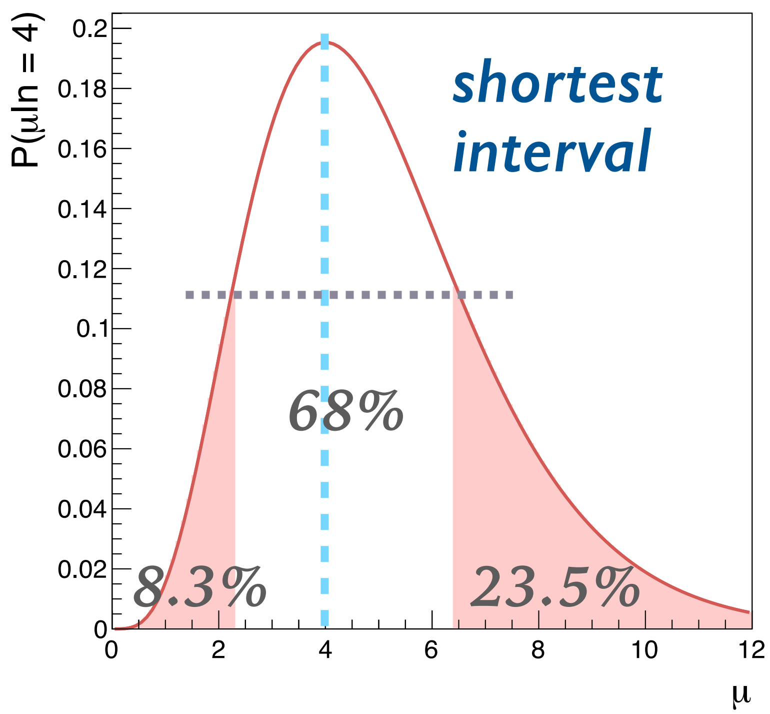
- Remember: one can claim that all the knowledge about the parameter is already **summarized in the posterior density** $\boldsymbol{P(\theta|X)}$ in the Bayesian parameter estimation.
- To compute an interval $[\theta_L, \theta^U]$ which contains a given probability β, one has to find two points that fulfill the condition:

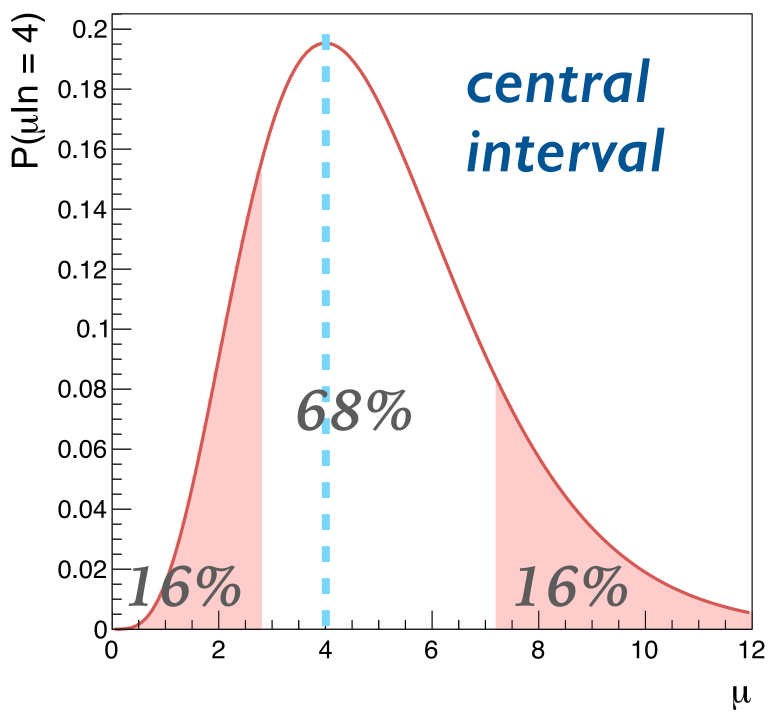
$$\int_{\theta_L}^{\theta^U} P(\theta|X)d\theta = \beta$$

- The usual choice of β either 68.3% or 90%. This represents the degree of belief that the true value of θ lies within the given range.
- A Bayesian interval with probability β is named as a credible interval, while the frequentist version is called the **confidence interval**.

INTERVAL ESTIMATION: BAYESIAN (CONT.)

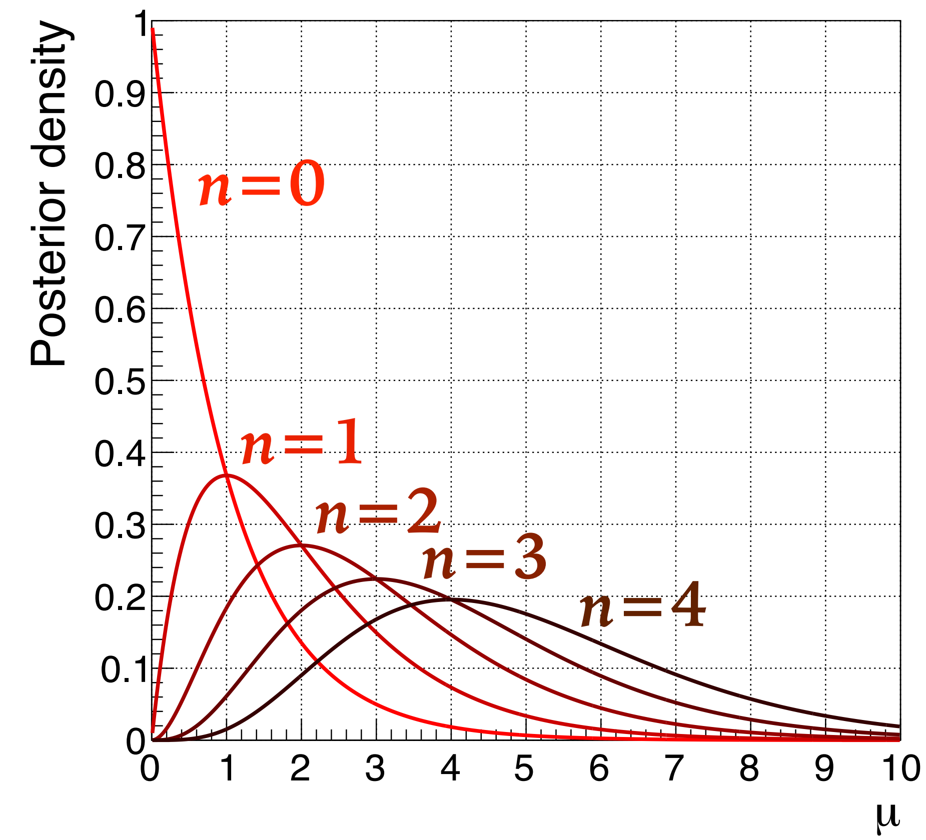
- Just with the given probability β, the construction of the credible interval is not unique. It is common to impose additional choice:
  - **Shortest interval**: integrate over the region with highest posterior density, until it reaches the given β. This interval is not invariant under parameter transformation.
  - **Central interval**: such that the integral of central part is β and in each side is $(1-\beta)/2$. Central interval is invariant under parameter transformation.
  - **One-sided interval**: usually for obtaining an upper limit, especially the case when θ is near the end of the allowed region. One-sided intervals are invariant.





NEAR THE PHYSICAL BOUNDARIES

- As a good feature: the **prior is always zero in the non-physical region** in the Bayesian method, this drives **the credible interval must be in the allowed region**.
- However a measurement near the edge of the physical region **will be biased toward the (*interior*) physically allowed region**.
  - In general this is a result since the **credible interval represents the belief** itself.
  - But this also means that now we cannot distinguish the information from the actual measurement or from the prior.



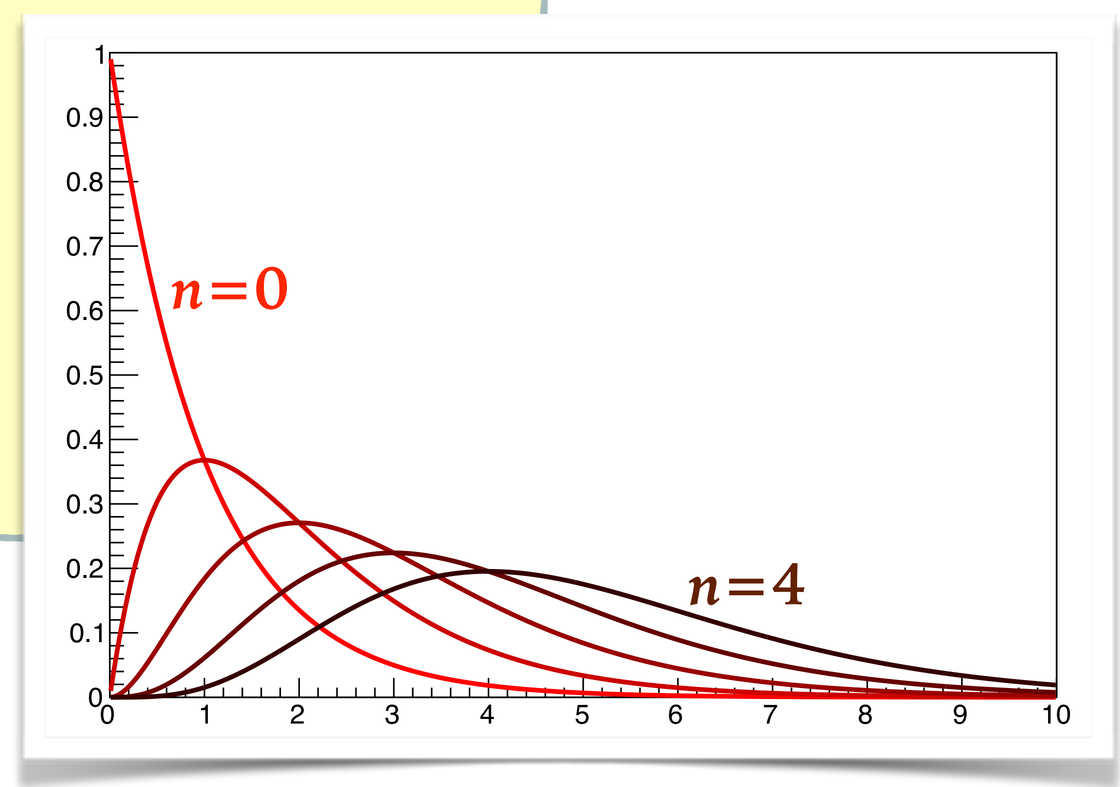
Bayesian posterior for Poisson with uniform prior

EXAMPLE: POISSON POSTERIOR WITH UNIFORM PRIOR

- Here we provide you an example code to produce the Bayesian posterior distributions for Poisson with uniform prior, with observed ***n*=0,1,2,3,4**:

example_01.cc

```
TH2D *frame = new TH2D(“frame","",10,0.,10.,10,0.,1.0);
frame->SetStats(false);
frame->Draw();

for (int n=0; n<=4; n++) {
    TH1D *P = new TH1D(Form("P%d",n),"",500,0.,10.);
    for (int i=1; i<=P->GetNbinsX(); i++) {
        double mu = P->GetBinCenter(i);
        P->SetBinContent(i,
           TMath::Poisson(n,mu));
    }
    P->SetLineWidth(3);
    P->SetLineColor(kRed+n);
    P->Draw("csame");
}
```

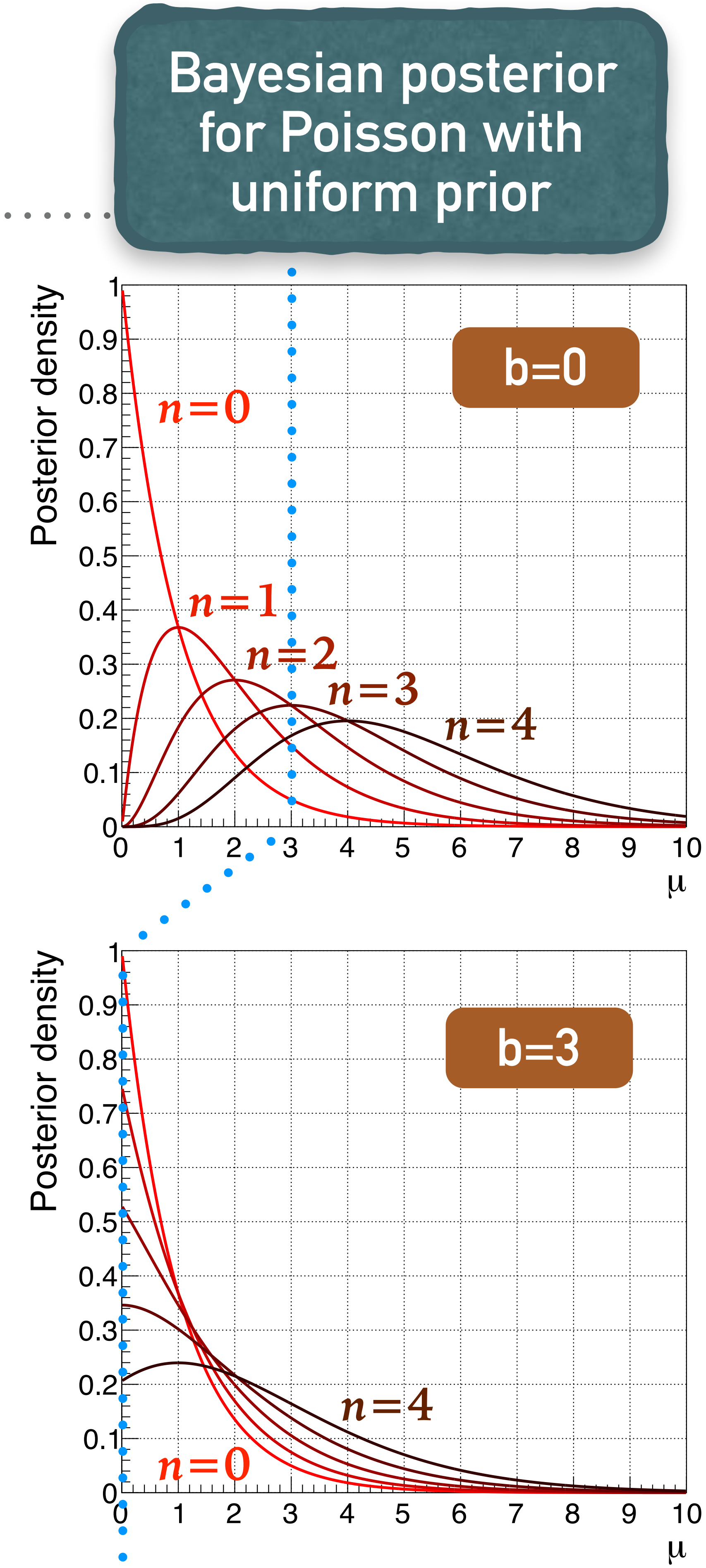


POISSON WITH BACKGROUND

One can separate the following two cases:

- **Exactly known background expectation**: the amount of background b is known and is fixed. e.g.
  - Expected $b=3.0$, observed $n=0$
  - Expected $b=2.5$, observed $n=8$
- **Background expectation is measured with some uncertainty**: the amount of background b has an error associated with it. e.g.
  - Expected $b=3.2\pm1.4$

  In this case b should be treated as a **nuisance parameter**.



EXPECTED BACKGROUND AS NUISANCE PARAMETER

- Everything has a probability distribution in the Bayesian framework, including the **nuisance parameters**!
- For example the distribution of background is described by a PDF $\boldsymbol{P(b)}$. In the calculation of the **posterior density**, one has to integrate over the nuisance parameters, e.g.

$$P(\theta|X) = \int_b \frac{P(X|\theta, b)P(\theta)}{P(X)} P(b)db$$

- This calculation might be heavy in terms of computing. However if background $\boldsymbol{b}$ is exactly known as $\boldsymbol{b_0}$, the $\boldsymbol{P(b)}$ is just a delta function and the integration becomes trivial.

$$P(\theta|X) = \int_b \frac{P(X|\theta, b)P(\theta)}{P(X)} \delta(b - b_0)db = \frac{P(X|\theta, b_0)P(\theta)}{P(X)}$$

EXAMPLE: POISSON POSTERIOR WITH NUISANCE

- Here are an example of calculating the Poisson posterior probability density with a nuisance parameter for background and is constrained by a Gaussian model.
- To make our life easier, the code is based on `RooStats::BayesianCalculator`. Surely you can also do all the calculation by yourself!

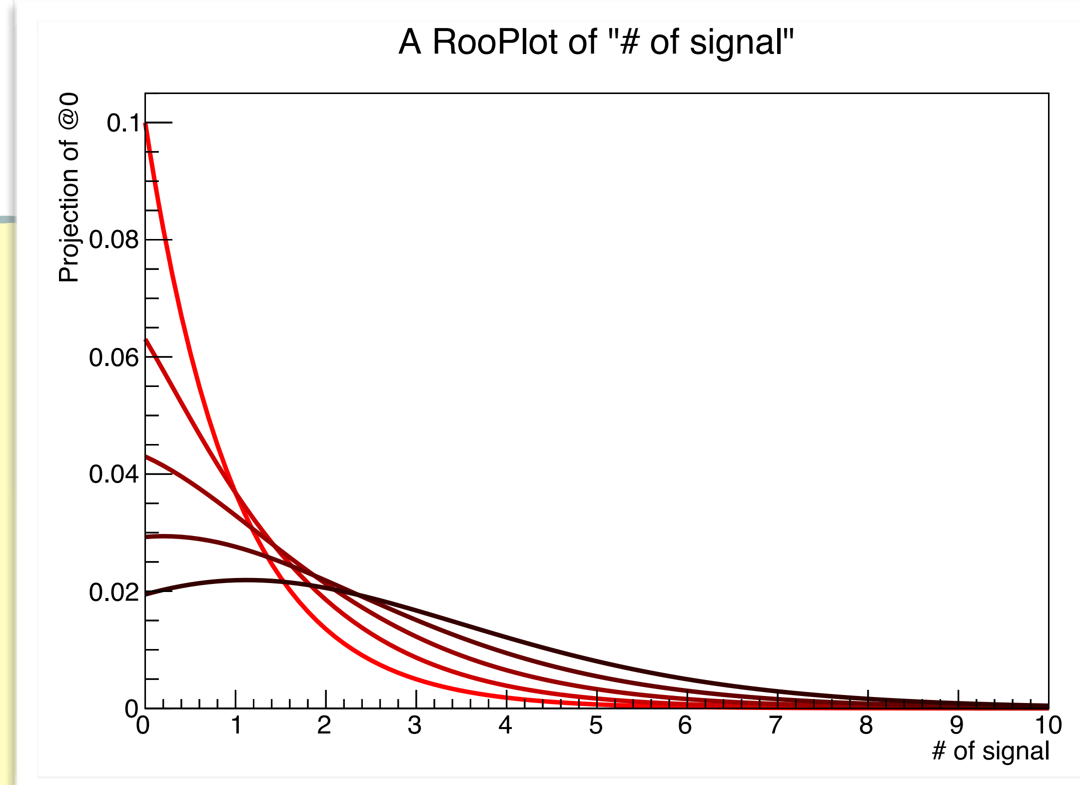
partial example_02.cc

```
using namespace RooFit;
using namespace RooStats;

RooRealVar x("x","dummy obs",0.,1.);
RooUniform pdf_x("pdf_x","dummy pdf of x",x);

RooRealVar s("s","# of signal",0.,10.);
RooRealVar b("b","# of background",3.2,0.,10.);
RooAddPdf pdf_splusb("pdf_splusb","total PDF",
    RooArgList(pdf_x,pdf_x),RooArgList(s,b));

RooUniform prior_s("prior_s","prior for signal",s);
RooGaussian prior_b("prior_b","priot for background",
    b,RooConst(3.2),RooConst(1.4));
RooProdPdf model("model","constrained model",pdf_splusb,prior_b);
```

Build a dummy extended s+b PDF for further use

times background prior (=constrained model)

EXAMPLE: POISSON POSTERIOR WITH NUISANCE (CONT.)



partial example_02.cc

put in the needed stuff!

```
RooWorkspace wspace("wspace");
ModelConfig cfg(&wspace);
cfg.SetPdf(model);
cfg.SetParametersOfInterest(s);
cfg.SetPriorPdf(prior_s);
cfg.SetNuisanceParameters(b);

RooPlot *frame = s.frame();
for (int n=0; n<=4; n++) {
    RooRealVar nevt("nevt","# of observed event",n);
    RooDataSet data("data","data",RooArgSet(x,nevt),WeightVar("nevt"));
    data.add(RooArgSet(x),n);

    BayesianCalculator bcalc(data,cfg);
    RooAbsPdf *posterior = bcalc.GetPosteriorPdf();
    posterior->plotOn(frame, LineColor(kRed+n),LineWidth(3));
}
frame->Draw();
```

weighted data set with dummy observable

POISSON WITH KNOWN BACKGROUND

- For the case of Poisson with known background and a **uniform prior**, the posterior density can be expressed as

$$P(\mu|n) \propto \frac{(\mu+b)^n}{n!} e^{-(\mu+b)}$$

- Let's practice the calculation of the 90% upper limit:

Observed	0	1	2	3
bkg = 0.0	2.30	3.89	5.32	6.68
0.5	2.30	3.51	4.84	6.18
1.0	2.30	3.27	4.44	5.71
2.0	2.30	2.99	3.88	4.93
3.0	2.30	2.84	3.52	4.36

You may find the results looks quite reasonable, no matter with or with our the background.
In particular when n=0, the limits decouple from the background.

- However a **uniform prior PDF $P(\mu)$ cannot represent the belief:**

$$\int_a^b P(\mu)d\mu = 0$$

this will happen for **any finite [*a*,*b*]**, since ***P*(*μ*)** has to be normalized in **[0,∞]**

THEN…TRY SOMETHING ELSE?

- Since the uniform prior has such a "brief" problem, let's examine the case of Jeffreys priors.
- Recall: Jeffreys priors are derived to be invariant under coordinate transformations.
- In particular **the prior $1/\mu$** is scale-invariant; it could represent the belief, since it goes to zero at infinity. But this does not work either since it goes to infinity when $\mu=0$.

Bayesian 90% upper limit with 1/μ prior

Observed	0	1	2	3
bkg = 0.0	0.00	2.30	3.89	5.32
0.5	0.00	0.00	0.00	0.00
1.0	0.00	0.00	0.00	0.00
2.0	0.00	0.00	0.00	0.00
3.0	0.00	0.00	0.00	0.00

The upper limits goes to zero since the posterior density goes to infinity when μ approaches zero.

THEN…TRY SOMETHING ELSE? (CONT.)

- There is another Jeffreys prior that minimizes the Fisher information contained in the prior:

$$P(\mu) = 1/\sqrt{\mu}$$

  but obviously this does not solve the divergences problem with background.

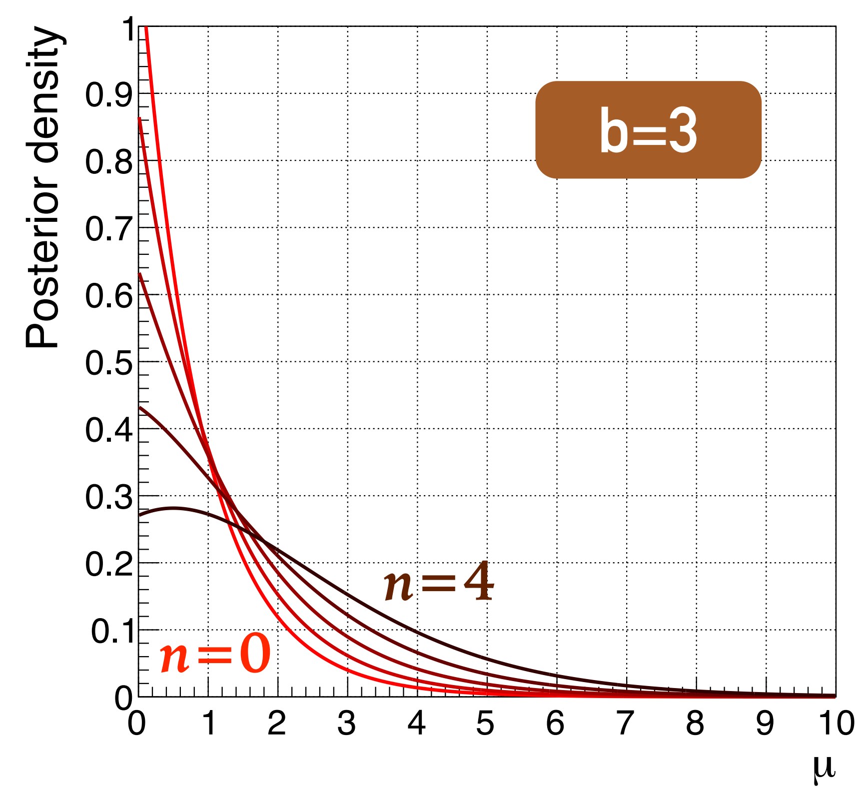
- The suggested Jeffreys prior to solve the case of Poisson with the expected background ***b*** is

$$P(\mu) = 1/\sqrt{\mu + b}$$

  but this also means **the prior has to depend on the background level**.

Bayesian 90% upper limit with prior $1/\sqrt{\mu + b}$

Observed	0	1	2	3
bkg = 0.0	1.35	3.13	4.62	6.01
0.5	1.81	2.88	4.17	5.52
1.0	1.92	2.76	3.84	5.07
2.0	2.03	2.63	3.41	4.38
3.0	2.08	2.55	3.16	3.92



EXAMPLE: POISSON WITH KNOWN BACKGROUND

- Here are an example of calculating the upper limit table for the case of Poisson with **known background** and a **uniform prior**. Again we are using `RooStats::BayesianCalculator` here.

partial example_03.cc

```
RooMsgService::instance().setGlobalKillBelow(RooFit::FATAL);

TCanvas *c1 = new TCanvas("c1","",800,600);
c1->SetMargin(0.05,0.05,0.22,0.05);
c1->Divide(4,4,0.,0.);

printf("\nObserved    0    1    2    3\n");
for (int b_set=0; b_set<4; b_set++) {
    printf("bkg = %d  ",b_set);
    for (int n_set=0; n_set<4; n_set++) {

        RooRealVar x("x","dummy obs",0.,1.);
        RooUniform pdf_x("pdf_x","dummy pdf of x",x);

        RooRealVar s("s","# of signal",1E-5,15.);
        RooRealVar b("b","# of background",b_set);
        RooAddPdf model("model","total PDF”,
            RooArgList(pdf_x,pdf_x),RooArgList(s,b));
        RooUniform prior_s("prior_s","prior for signal",s);
```

Build a dummy extended PDF

EXAMPLE: POISSON WITH KNOWN BACKGROUND (II)

partial example_03.cc

```
        RooWorkspace wspace("wspace");
        ModelConfig cfg(&wspace);
        cfg.SetPdf(model);
        cfg.SetParametersOfInterest(s);
        cfg.SetPriorPdf(prior_s);

        RooRealVar nevt("nevt","# of event",n_set);
        RooDataSet data(“data","data",
            RooArgSet(x,nevt),WeightVar("nevt"));
        data.add(RooArgSet(x),n_set);

        BayesianCalculator bcalc(data,cfg);
        bcalc.SetLeftSideTailFraction(0.);
        bcalc.SetConfidenceLevel(0.9);
        SimpleInterval* interval = bcalc.GetInterval();
        printf("%.2f ",interval->UpperLimit());

        c1->cd(b_set*4+n_set+1);
        RooPlot *frame = bcalc.GetPosteriorPlot();
        frame->Draw();
    }
    printf("\n");
}
```

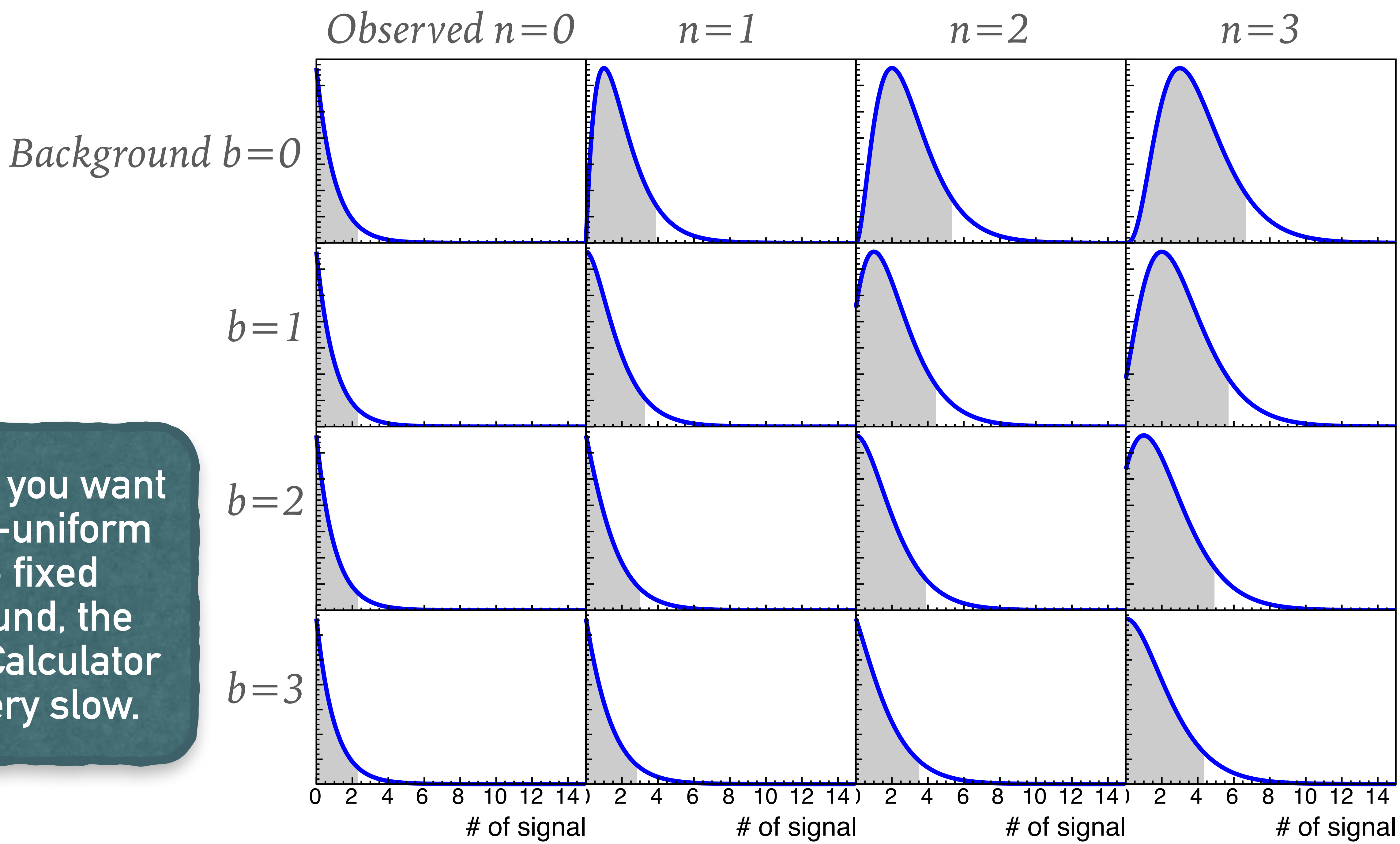
Observed	0	1	2	3
bkg = 0	2.30	3.89	5.32	6.68
bkg = 1	2.30	3.27	4.44	5.71
bkg = 2	2.30	2.99	3.88	4.93
bkg = 3	2.30	2.84	3.52	4.36

Ask the calculator to evaluate 1-side upper limit

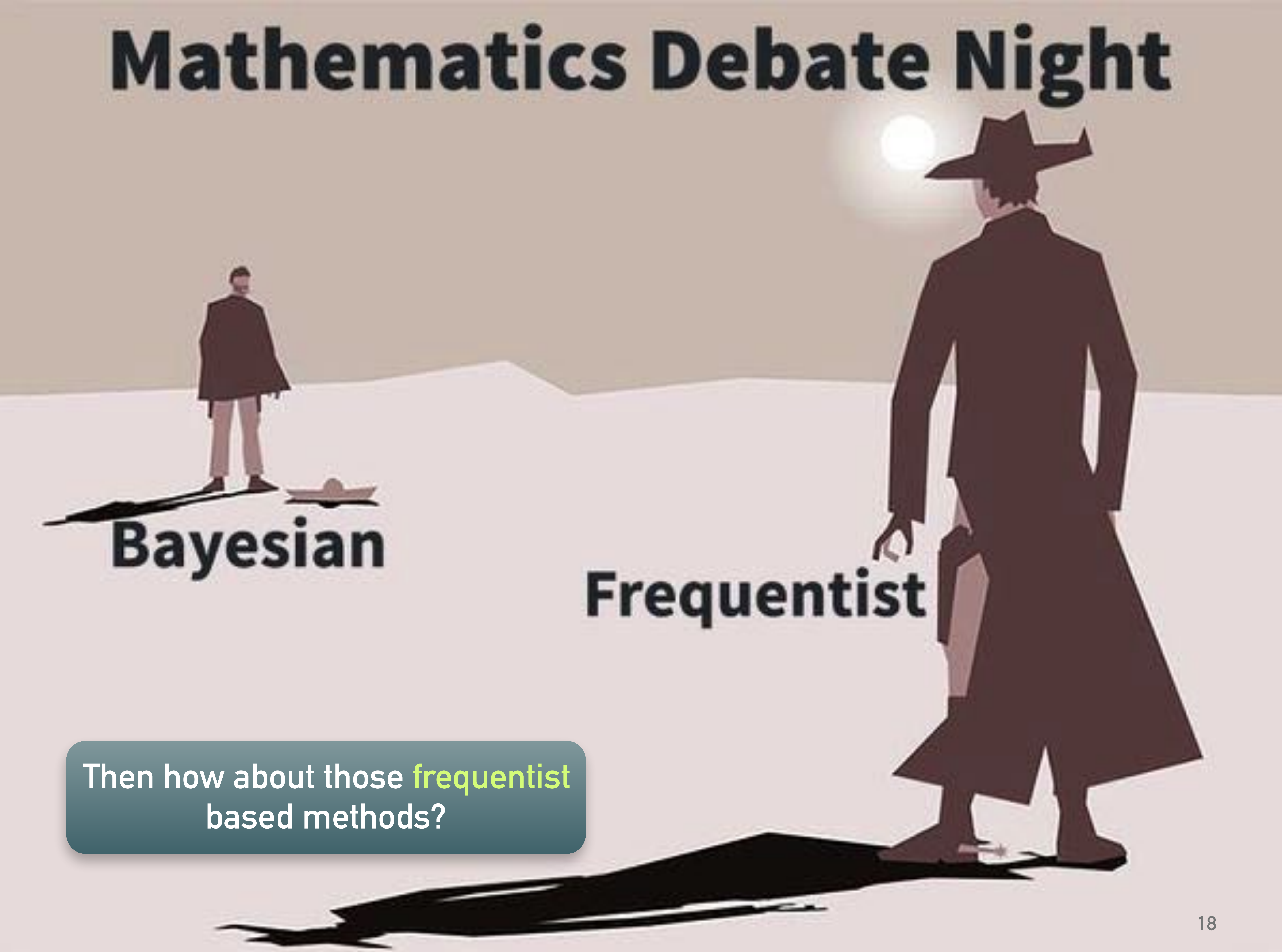
You are encouraged to play with different priors & adding background nuisance PDF!

EXAMPLE: POISSON WITH KNOWN BACKGROUND (III)

➤ Also get the posterior density for each configuration:



Remark: if you want to try non-uniform prior + fixed background, the BayesianCalculator will be very slow.



Then how about those frequentist based methods?

INTERVAL ESTIMATION: FREQUENTIST

- The central problem: given a probability β, find the optimal range $[\theta_a,\theta_b]$ in the space of θ and:

$$P(\theta_a \leq \theta_0 \leq \theta_b) = \beta$$

where θ_0 is the true value of θ. The interval (θ_a,θ_b) is a **confidence interval**.

- A method gives an intervals (θ_a,θ_b) satisfying the equation above is said to have the **property of coverage**. Note: the (θ_a,θ_b) are random variables, θ_0 is not.

- If an interval does not hold the property of coverage, it cannot be a confidence interval in fact. Although one can still consider **approximate confidence intervals**, which have only approximate coverage.
  - **Over-coverage** means when $P > \beta$.
  - **Under-coverage** means when $P < \beta$ (*bad!*)

NORMAL THEORY INTERVAL ESTIMATION

- A data ***X*** is sampling from a Gaussian distribution ***N(μ,σ)***.
- The relation between the given probability and the interval is straightforward when both the mean ***μ*** and variance ***σ²*** are known:

$$\beta = P(a \leq X \leq b) = \int_a^b N(X;\mu,\sigma)dX$$

- When ***μ*** is unknown, the probability content of the interval **[*a*, *b*]** cannot be calculated anymore. But it is possible to evaluate the probability ***β*** where ***X*** lies in some interval **relative to its unknown mean**, e.g. ***[μ+c , μ+d]***, with a simple transformation ***Y=(X–μ)/σ***:

$$\beta = P(\mu + c \leq X \leq \mu + d) = \int_{\mu+c}^{\mu+d} N(X;\mu,\sigma)dX = \int_{c/\sigma}^{d/\sigma} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}Y^2\right) dY$$

*(no **μ** here!)*

- Re-arrange the inequalities inside the probability gives:

$$\beta = P(\mu + c \leq X \leq \mu + d) = P(X - d \leq \mu \leq X - c)$$

NORMAL THEORY INTERVAL ESTIMATION (II)

- This is working since (*by shifting the calculation of probability β around the given data* X):
  - The data are distributed according to Gaussian, and the Gaussian PDF is **symmetric** in X and μ, since it is a function only of $(X-\mu)^2$.
  - We have assumed that the integration of both variables can be carried out in both directions, as far as they go, i.e., there is **no colliding of physical boundaries**.
- According to the theory of point estimation discussed in the previous lecture, both of the above bullets are true for **the maximum likelihood and least squares estimators**, asymptotically (valid when $N\to\infty$).
- So we already have **an asymptotically working theory of interval estimation**!

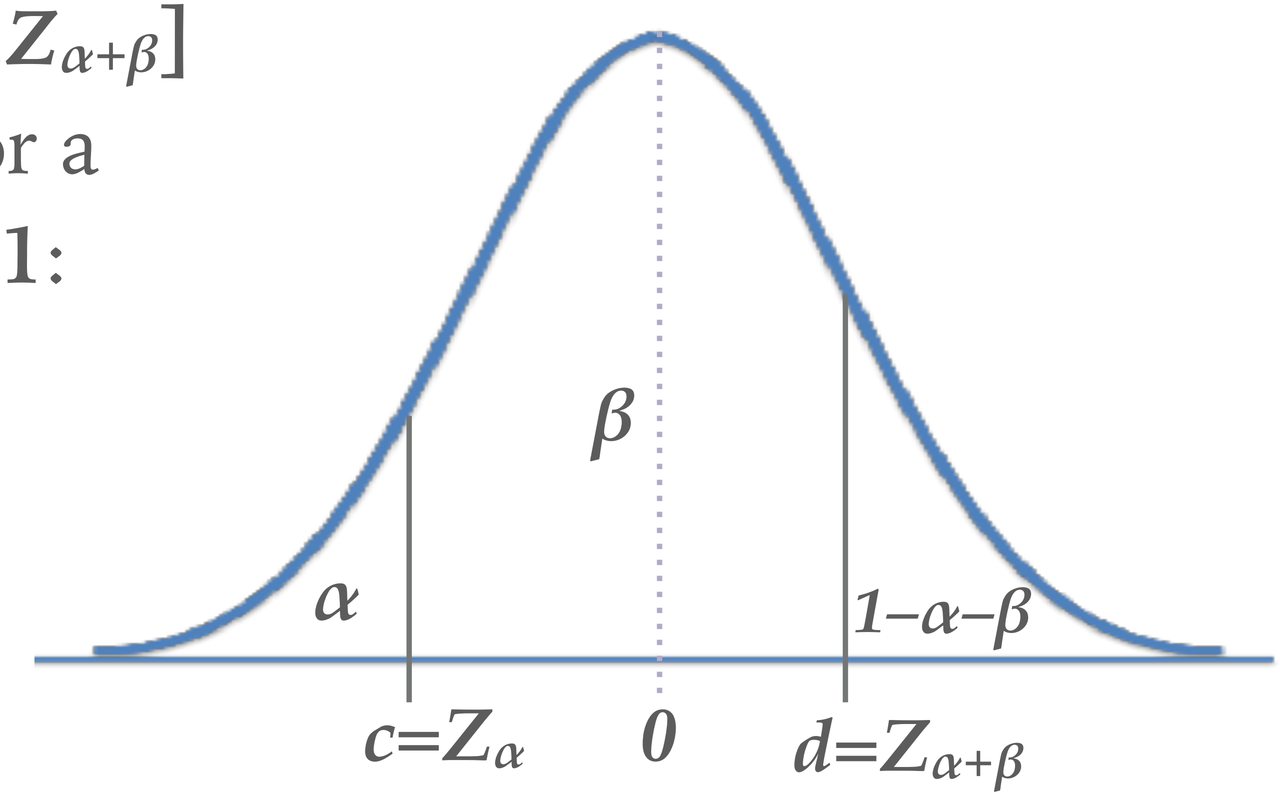
The extension to multiple variables is also straightforward!

NORMAL THEORY INTERVAL ESTIMATION (III)

- This is something very similar to what we already discussed when introducing Gaussian distribution itself!
- Given a random variable $\boldsymbol{X}$ with Gaussian PDF $\boldsymbol{f(X)}$ and the cumulative distribution $\boldsymbol{F(X)}$, the "$\boldsymbol{\alpha}$-point" $\boldsymbol{X_\alpha}$ is defined by

$$\int_{-\infty}^{X_\alpha} f(X)dX = F(X_\alpha) = \alpha$$

- The interval $\mathbf{[c,d]}$ is just $\mathbf{[Z_\alpha, Z_{\alpha+\beta}]}$ obviously, as shown below for a Gaussian PDF with $\boldsymbol{\mu=0, \sigma=1}$:



NORMAL THEORY INTERVAL ESTIMATION (IV)

- For a given value of β, there are many possible choice of value α, and hence there are multiple intervals as a result.
- Surely the most common choice is $\alpha=(1-\beta)/2$, which gives the **central interval**, and is symmetric around zero.
- This is what we have seen before, for a standard Normal distribution:

$\beta=(1-\alpha)/2$	Z_α	$Z_{\alpha+\beta}$
0.6827	-1.00	+1.00
0.9000	-1.65	+1.65
0.9500	-1.96	+1.96
0.9545	-2.00	+2.00
0.9900	-2.58	+2.58
0.9973	-3.00	+3.00

INTERVAL IN MULTIPLE VARIABLES

- In the case of more than one dimension, the Gaussian PDF can be expressed as (*note we have discussed this in the earlier lecture once!*):

$$f(X) \propto \exp\left[-\frac{1}{2}(X-\mu)^T \cdot V^{-1} \cdot (X-\mu)\right] = \exp\left(-\frac{1}{2}Q\right)$$

- The quantity $\boldsymbol{Q=(X-\mu)^T V^{-1}(X-\mu)}$ is the **covariance form** of X, and it follows a **χ^2 distribution with k degrees of freedom**. This means **Q** is not really dependent on $\boldsymbol{\mu}$, we can use χ^2 distribution to define the $\boldsymbol{\beta}$ point ($\boldsymbol{K_\beta{}^2}$):

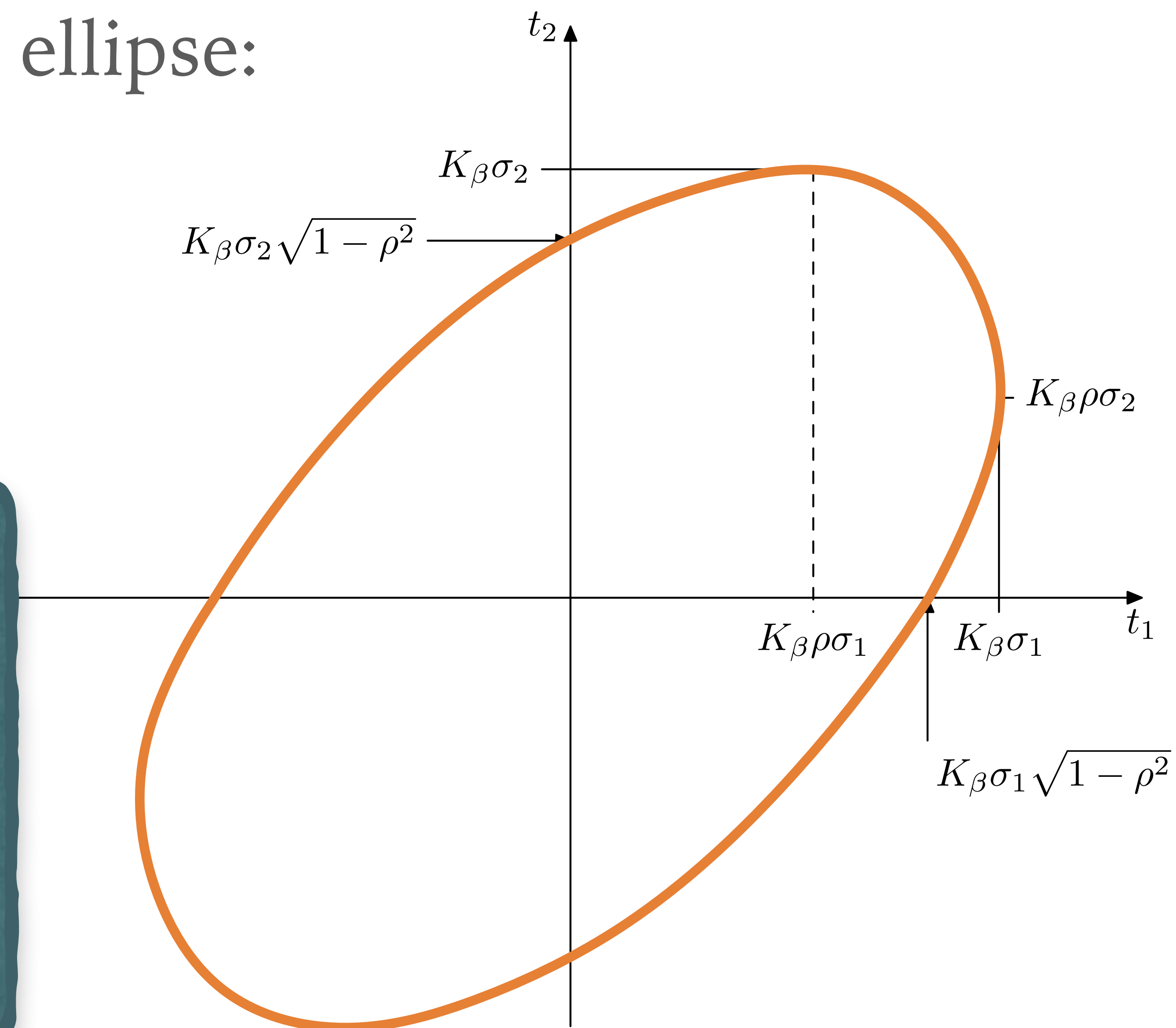
$$P[Q(X,\mu) \leq K_\beta^2] = \beta$$

- Thus the confidence interval becomes a **confidence region** with probability content $\boldsymbol{\beta}$, defined by $\boldsymbol{Q \leq K_\beta{}^2}$.

CONFIDENCE REGION IN 2D

- Consider two Normally distributed variables (t_1,t_2) with the corresponding covariance matrix V (which contains the correlation parameter ρ).
- This is just the standard error ellipse:

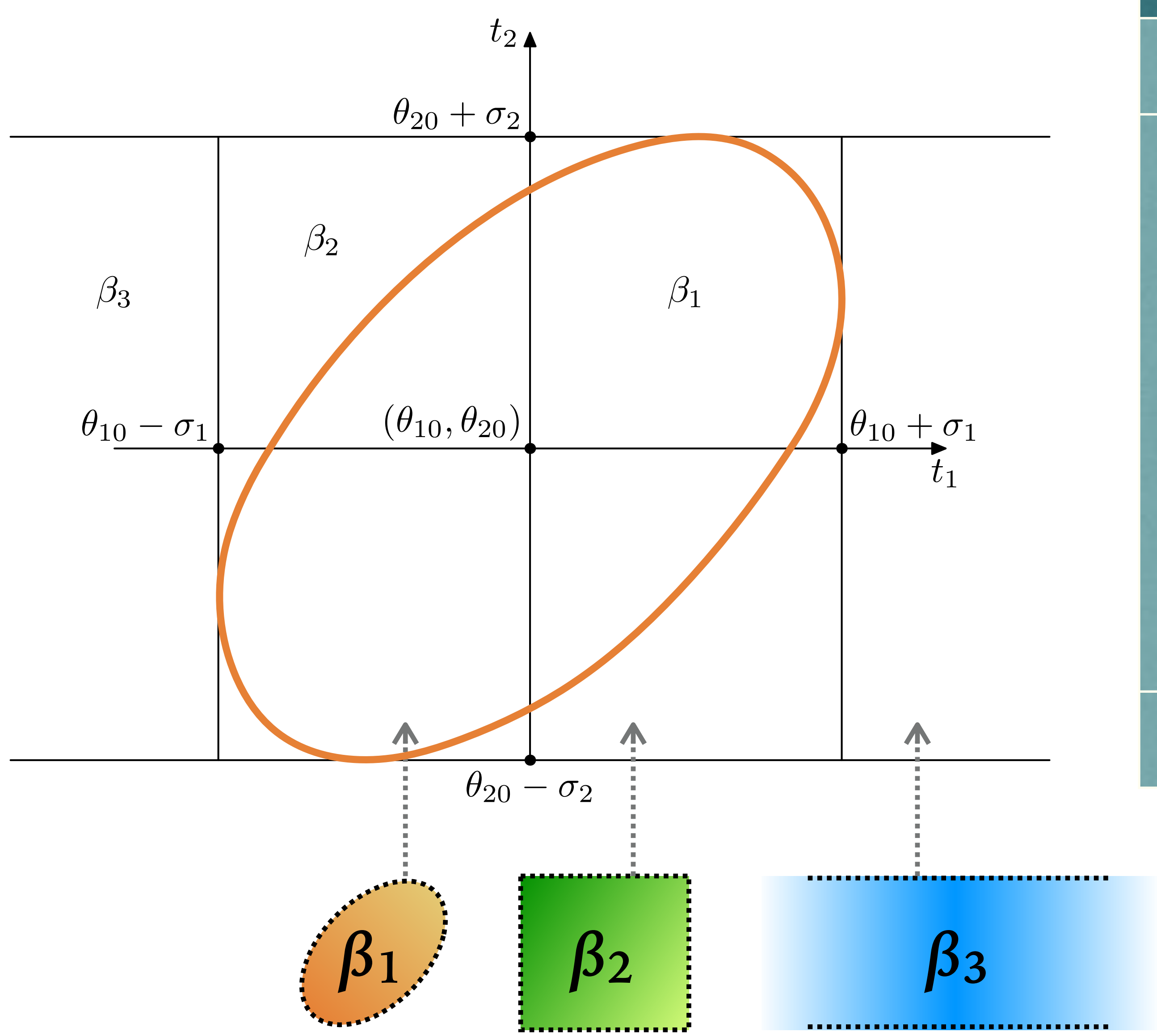
$$V = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$



- If ρ is positive (negative), the major axis of the ellipse has a slope of +1(–1).
- If $\rho=0$, the axes of the ellipse coincide with the coordinate axes.
- If $\rho=1$, the ellipse degenerates to a diagonal line.

CONFIDENCE REGION IN 2D (CONT.)

- For two Gaussian-distributed variables, the probability contents of the three regions for different values of K_β and different correlation ρ.



		$K_\beta=1$	$K_\beta=2$	$K_\beta=3$
β_1		0.393	0.865	0.989
β_2	$\rho=0.00$	0.466	0.911	0.995
	$\rho=0.50$	0.498	0.917	0.995
	$\rho=0.80$	0.561	0.929	0.996
	$\rho=0.90$	0.596	0.936	0.996
	$\rho=0.95$	0.622	0.941	0.996
	$\rho=1.00$	0.683	0.954	0.997
β_3		0.683	0.954	0.997

β_1 & β_3 are independent of ρ!

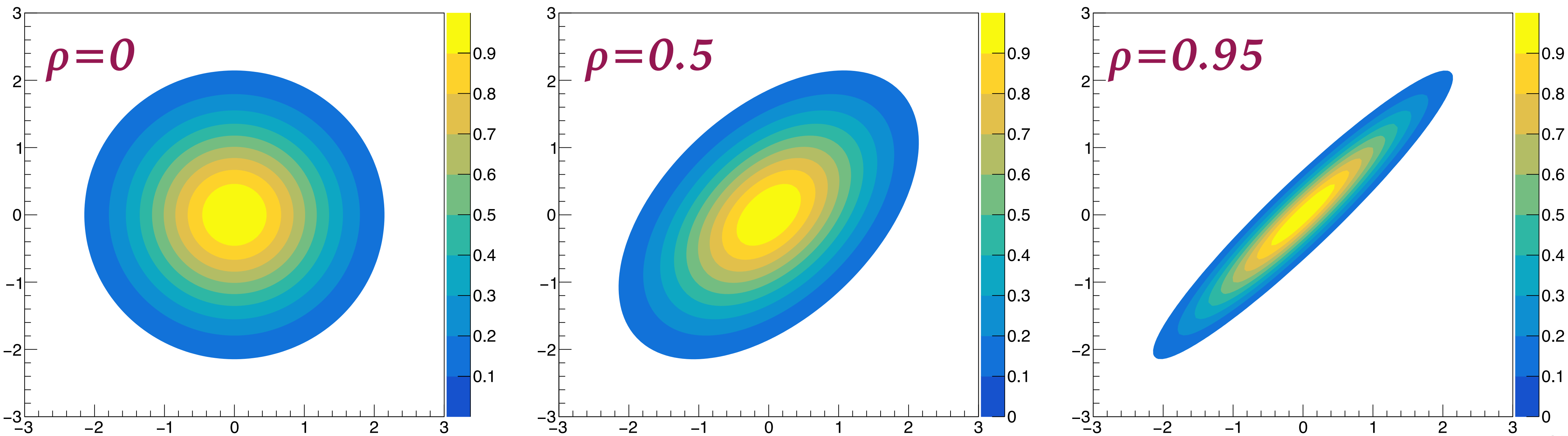
EXAMPLE: DRAWING THE CORRELATED 2D GAUSSIAN

➤ Here we try to draw the contours with a 2D correlated Gaussian distribution, as a straightforward demonstration of the correlation ρ!

example_04.cc

```
TF2 *func = new TF2("func","exp(-0.5*(x*x+y*y-2*[0]*x*y)/(1-[0]*[0]))",-3,3,-3,3);
func->SetParameter(0,0.9);
func->SetContour(10);
func->SetNpx(100);
func->SetNpy(100);
func->Draw("cont0z");
```

↶Put the ρ value here!

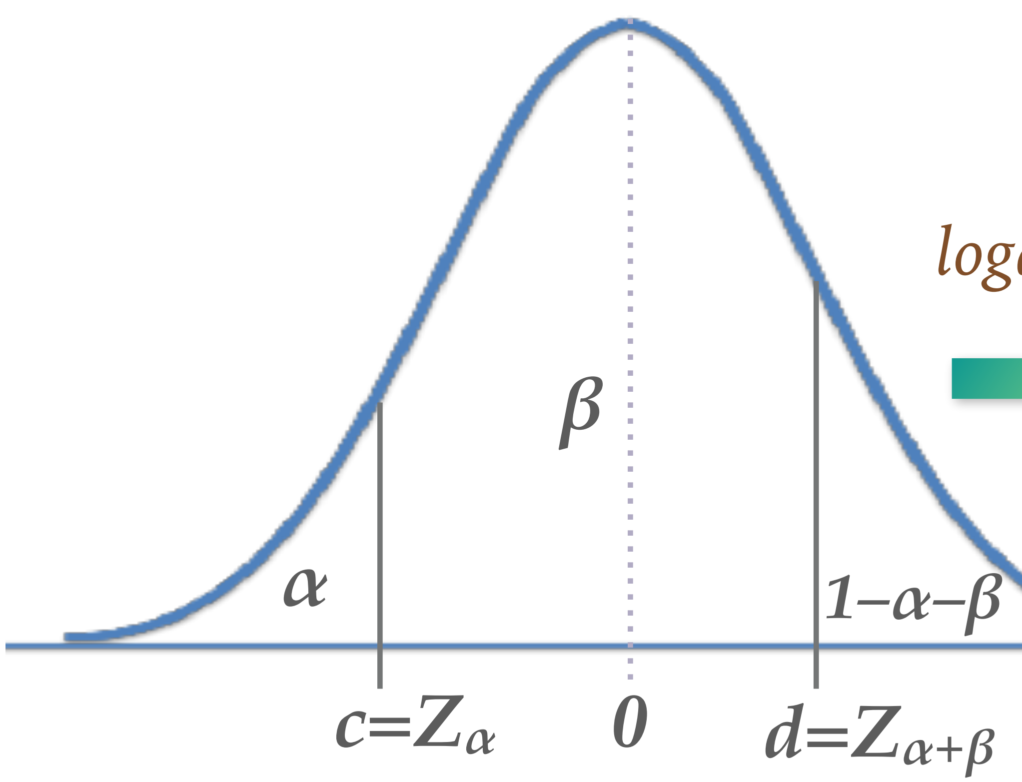


LIKELIHOOD-BASED CONFIDENCE INTERVALS

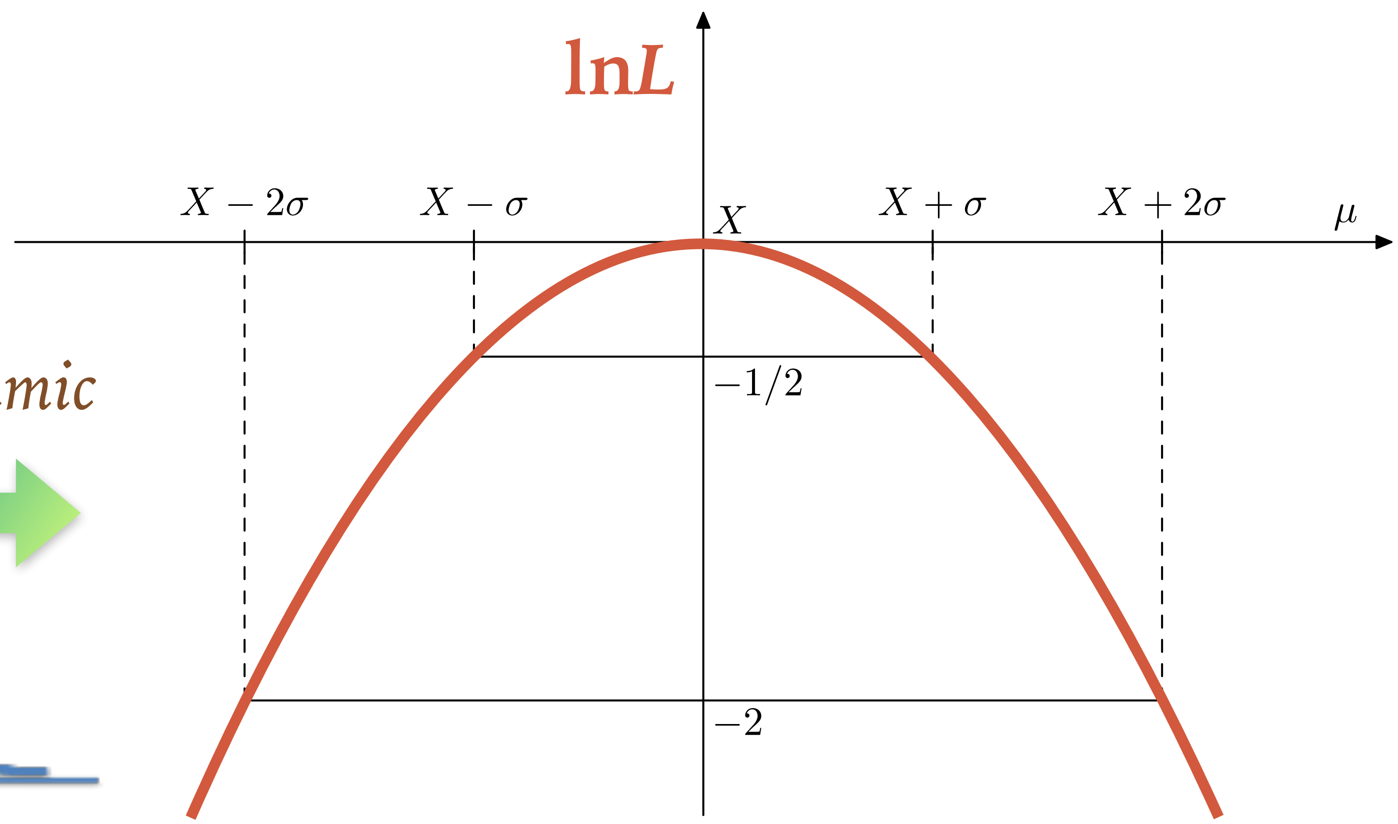
- Now we quickly revisit the method we already touched in the previous lecture: **likelihood-based confidence intervals**. It is kind of intermediate method between the Normal theory intervals (*discussed above!*) and the exact frequentist method (*to be discussed afterwards*).
- This method was suggested by D. Hudson, a statistician working at CERN in 1964 –– it was implemented already in Minuit since 1966! At the time, the properties of this method were not fully understood and only works for simple (single-parameter) problems.
- This is exactly the **"MINOS"** method, as the Minuit command which calculates this confidence interval. Statisticians started to study it for the multi-parameter case, ie. the **profile likelihood method**. It turns out to have a good coverage!

LIKELIHOOD-BASED CONFIDENCE INTERVALS (II)

➤ Recall the Normal Theory, we have showed how to convert the PDF into a likelihood function. Now if one takes the logarithm of the likelihood function, the Gaussian becomes a parabola.



Gaussian distributed N(μ,σ).



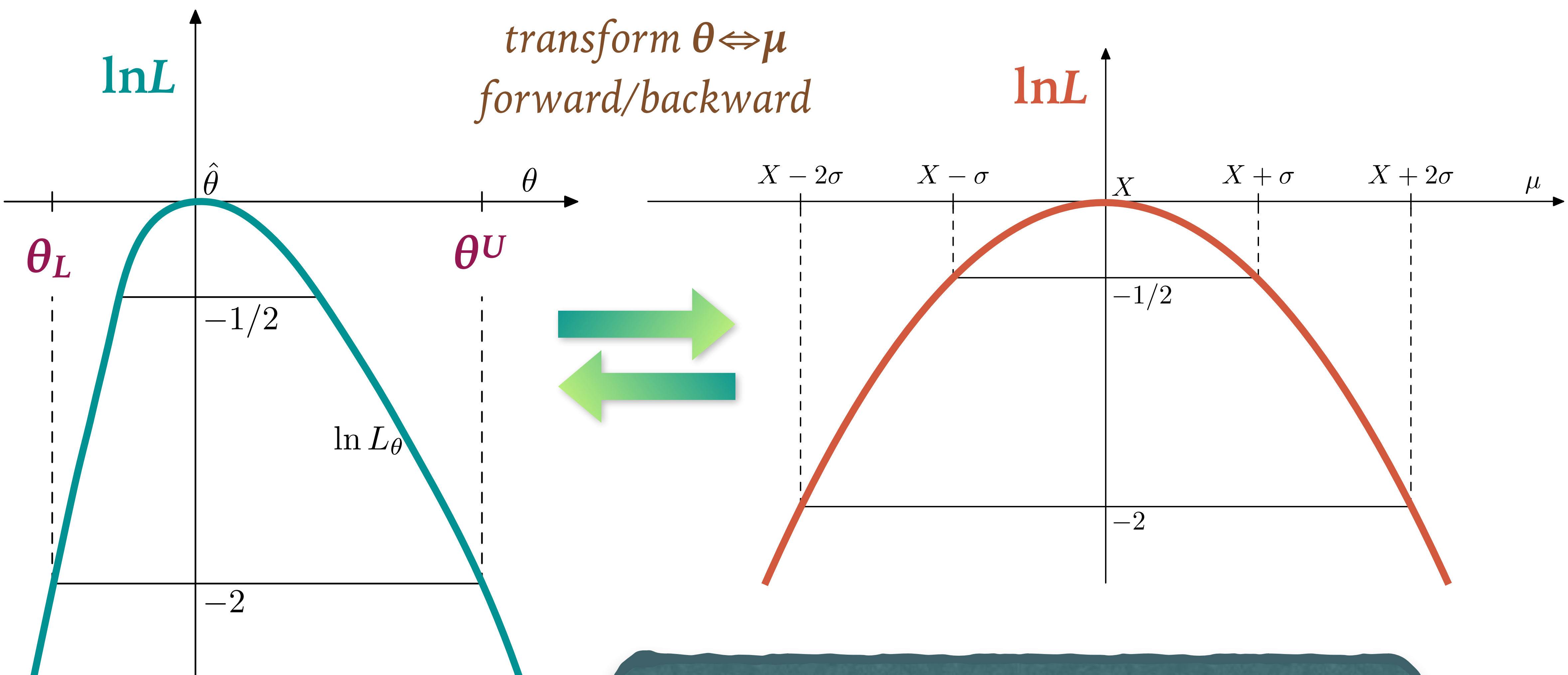
Log-likelihood function for Gaussian X, distributed N(μ,σ).

LIKELIHOOD-BASED CONFIDENCE INTERVALS (III)

- When the data are Gaussian-distributed, the Normal Theory can be applied. In this case the confidence intervals can be calculated easily, the log-likelihood would have an exact parabolic shape as shown before.
- Then how about an inverse case: **a parabolic log-likelihood function also implies that the Normal theory is applicable!**
  - Generally if the log-likelihood function is **not parabolic**, it can be (*mostly always*) transformed to a parabola by a **transformation** of the parameter ⇒ Normal theory applied.
  - However, the values of the likelihood are actually invariant under transformation, it is not necessary to find what is exactly the transformation, if we just need the likelihood itself.

LIKELIHOOD-BASED CONFIDENCE INTERVALS (IV)

- Since the likelihood values are invariant, one only needs to catch the parameter values when $\mathbf{2lnL=2lnL_{max}-1}$, for the standard one-sigma confidence interval.

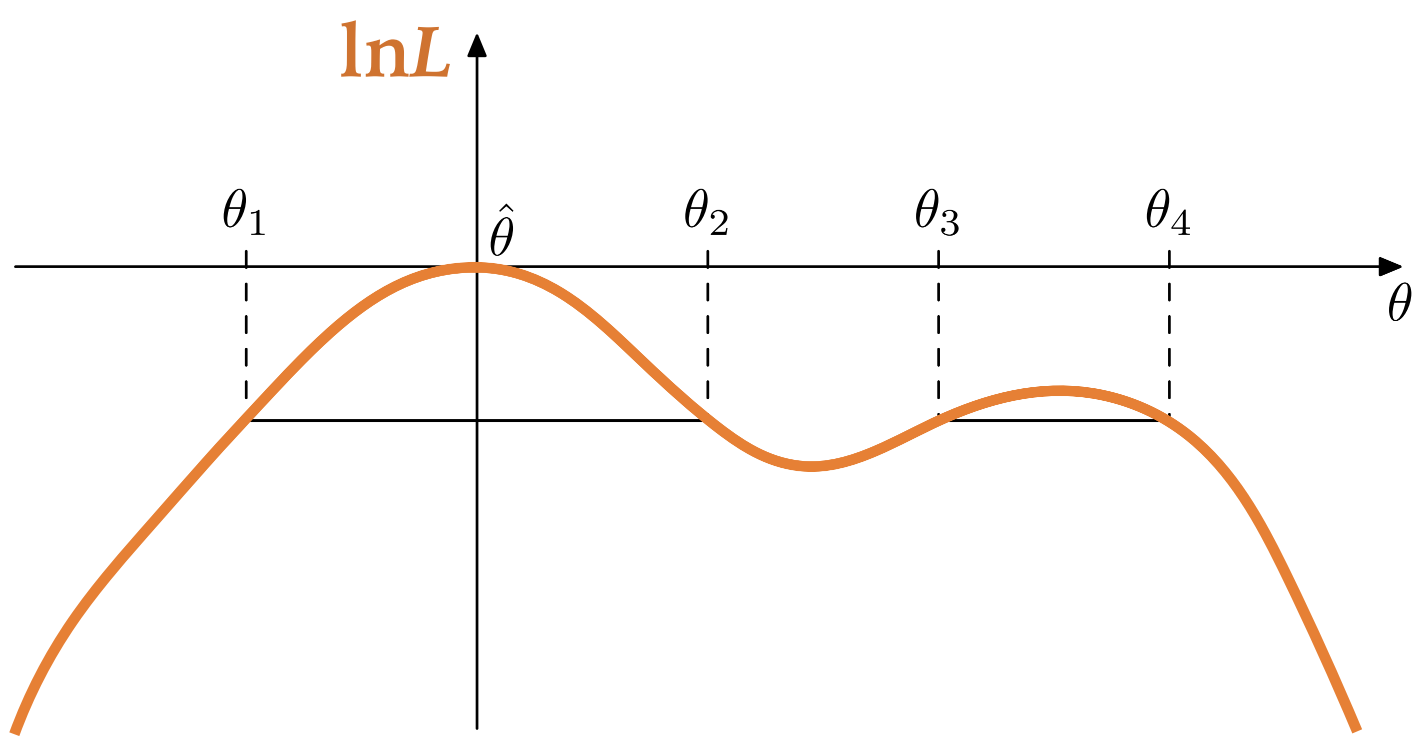


Non-parabolic log-likelihood

The interval $[\theta_L, \theta^U]$ are the likelihood values are basically invariant

LIKELIHOOD-BASED CONFIDENCE INTERVALS (V)

- In fact this method does not have good coverage for the simplest problem, i.e. Poisson distribution with very few events and no background, or some cases with "pathological" log-likelihood function, e.g.

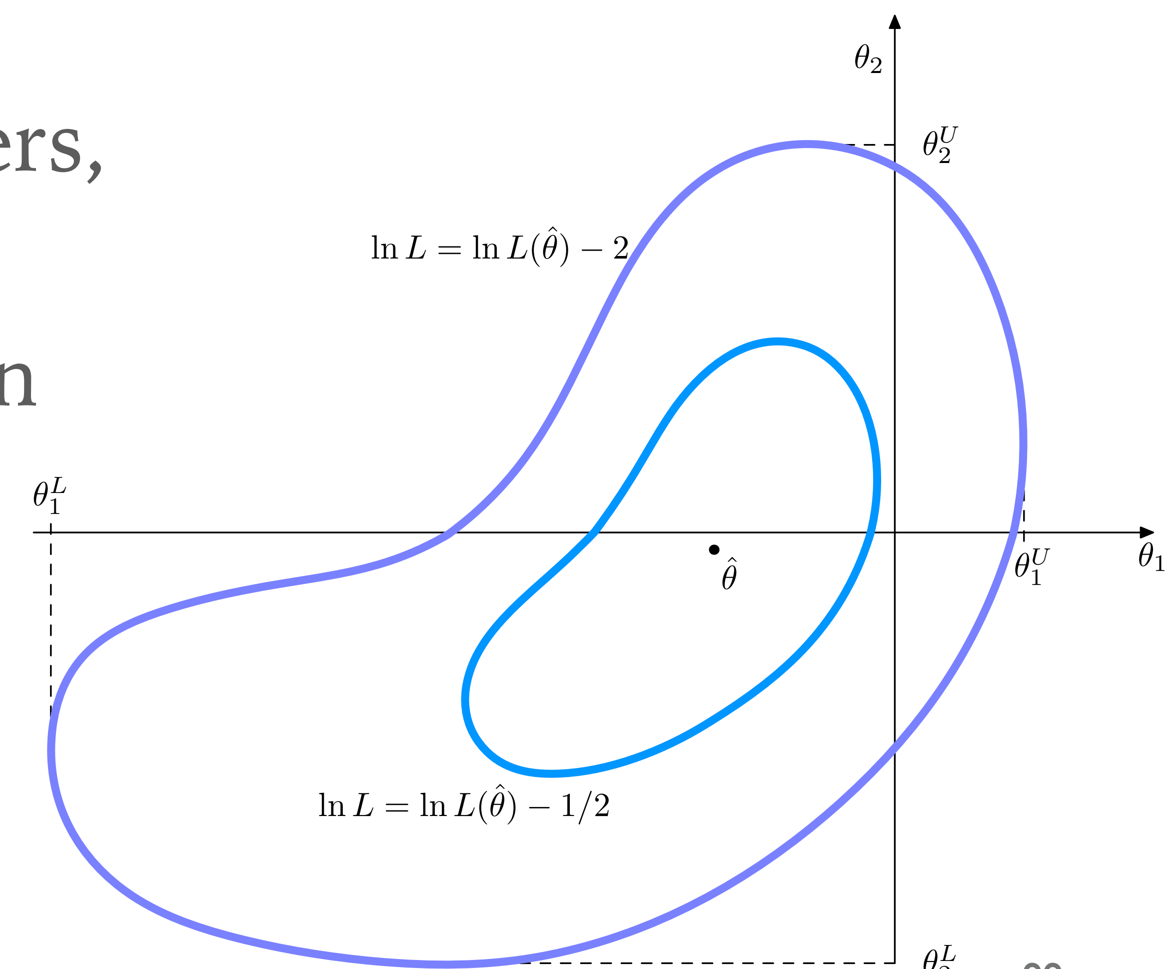


You will run into the problem of local minimum in this case.

- However this method turns out to have very good coverage for large numbers of parameters, so it is good for handling the **nuisance parameters**.

THE "MINOS" COMMAND

- The command MINOS in Minuit finds the intersection of the log-likelihood function with $2\ln L/L_{max}=1$. This gives an ***asymmetric confidence interval*** in general case.
- The Normal theory always gives a ***symmetric interval*** around the best the estimated value.
- When there are multiple parameters, the MINOS error is calculated by maximizing the likelihood function with respect to all other floated parameters, a.k.a. the **profile likelihood method**.



NEYMAN CONSTRUCTION OF CONFIDENCE INTERVALS

- Polish mathematician Jerzy Neyman, together with Egon Pearson produced two major contributions in statistics:
  - **The Neyman construction of confidence intervals**
  - **The Neyman-Pearson Test** (*to be discussed in the next lecture*)
- As for finding the exact frequentist intervals, the first important step is to work in the right space: **P(data|hypothesis)** — *with one axis for data, and another one for hypotheses*
  - In fact trying to place "true values" and "measured values" on the same axis is not a good approach, since hypotheses and data are living in different spaces.

NEYMAN CONSTRUCTION

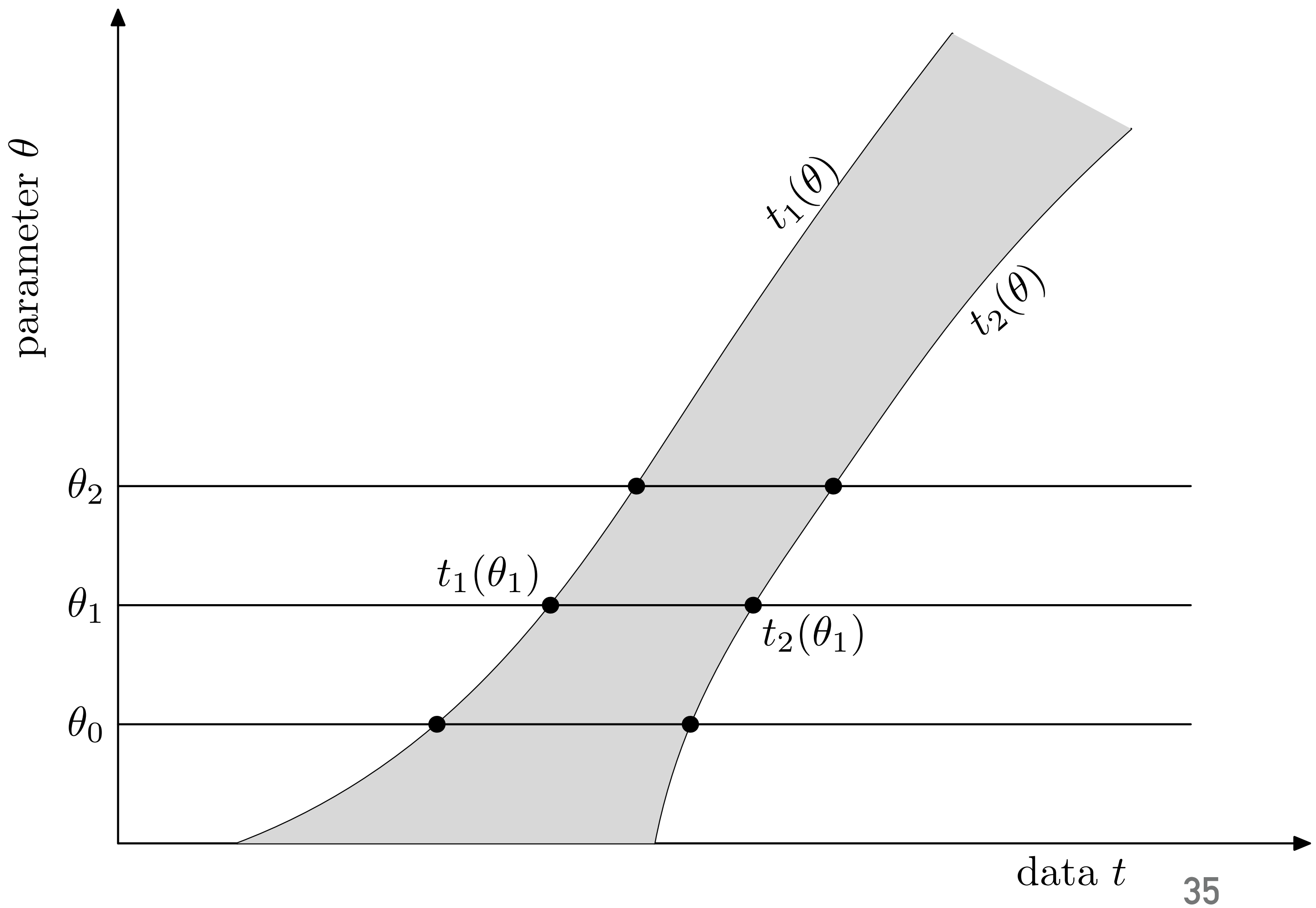
- Construct the confidence belt horizontally –– **for each hypothetical value of θ**, the points $t_1(\theta)$ and $t_2(\theta)$ are determined such that:

$$\int_{t_1}^{t_2} f(t|\theta)dt = \beta$$

where $f(t\,|\,\theta)$ is the PDF.

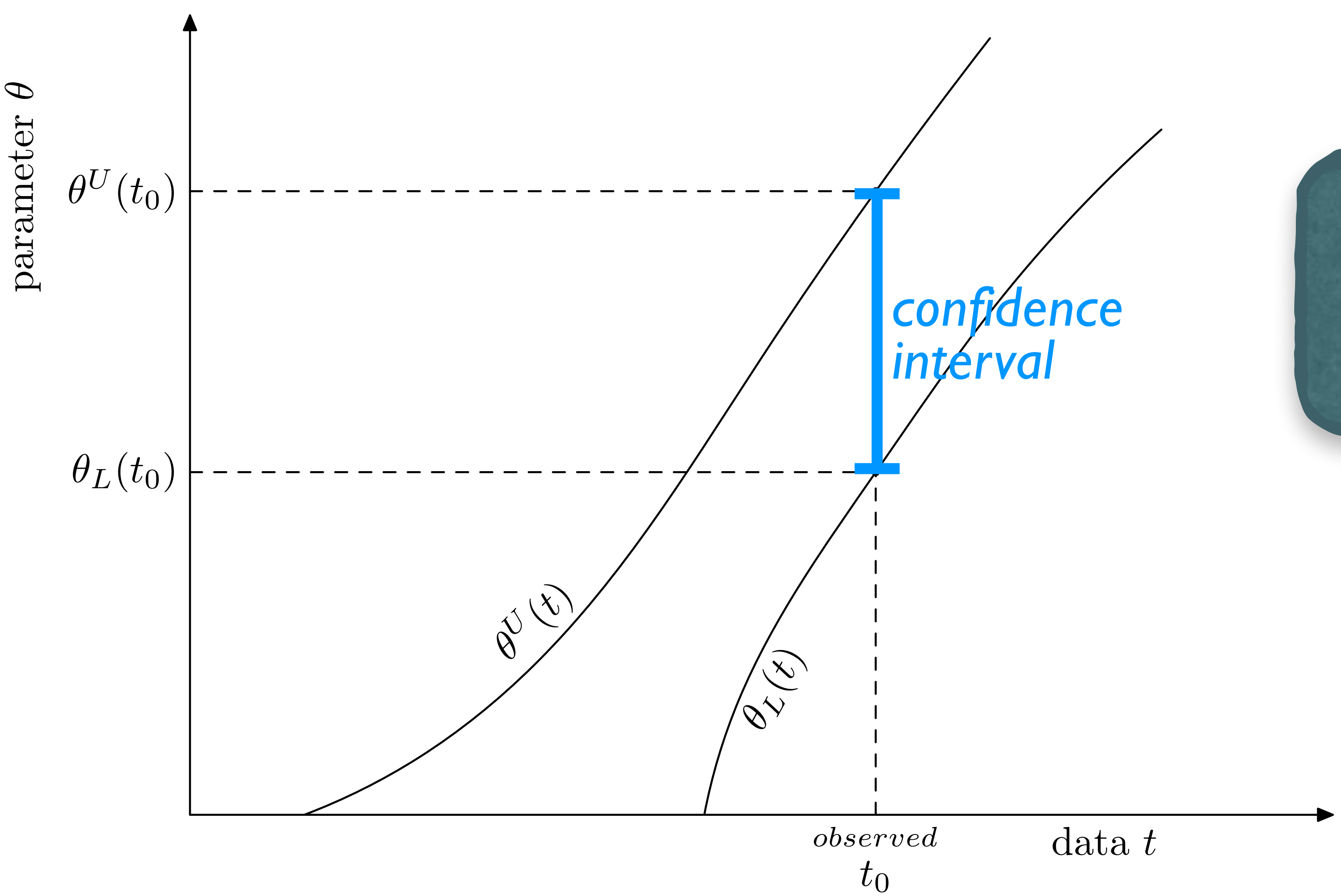
- Typical choices of β are 68.3% or 90%.
- Repeating the calculation for $t_1(\theta)$ and $t_2(\theta)$ and draw the confidence belt:

Remark: the solutions of t_1 and t_2 are not unique for each θ; here we just assume they can be obtained/defined using some way!
(e.g. central interval)



NEYMAN CONSTRUCTION (II)

➤ Now the two curves of $t_1(\theta)$ and $t_2(\theta)$ are re-labelled as $\theta(t)$, then the **confidence interval** can be read ***vertically***.

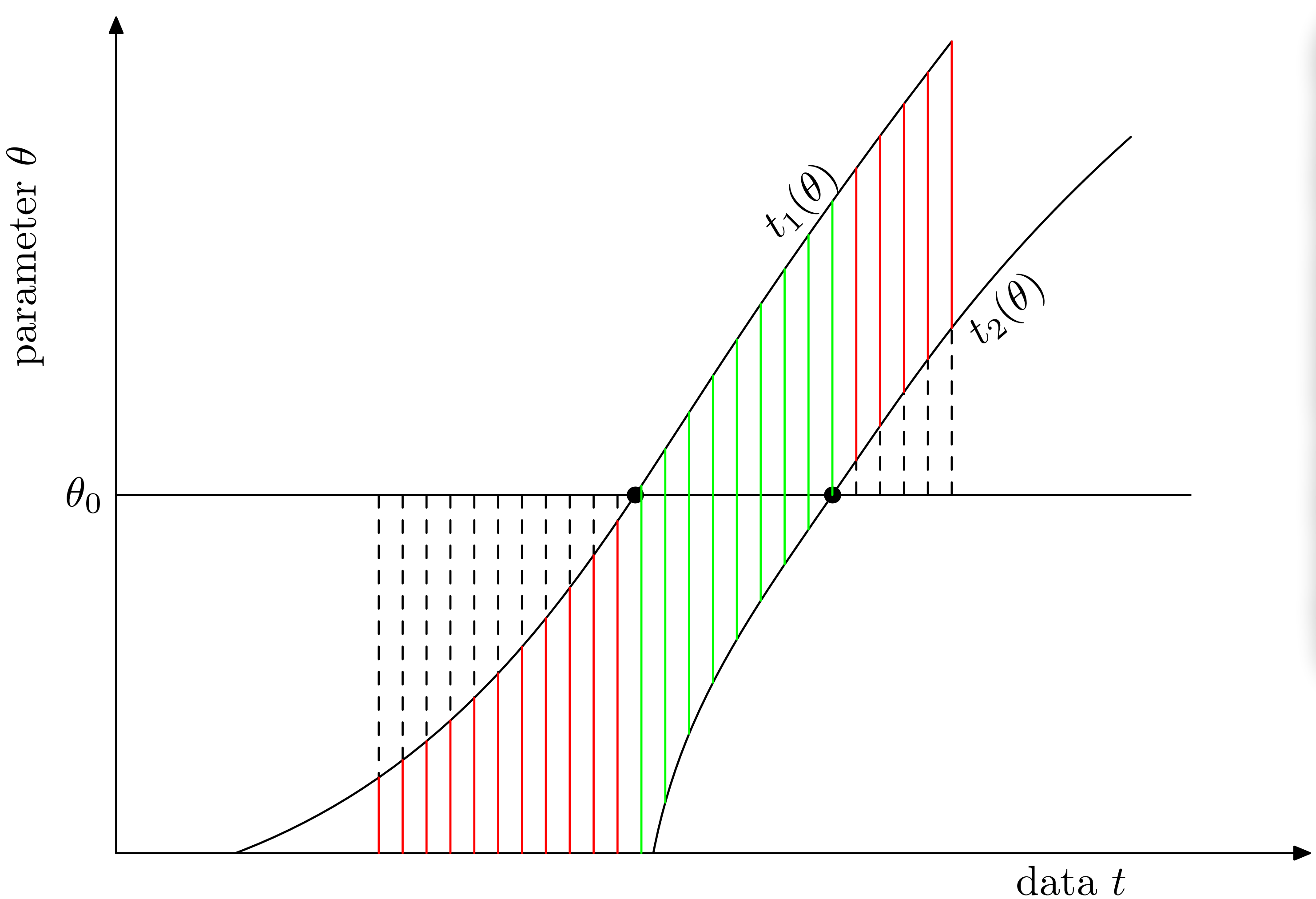


For given observed data t_0, the confidence interval is then $[\theta_L(t_0), \theta^U(t_0)]$.

Question: how could we claim the coverage, e.g. $P(\theta_L < \theta_{true} < \theta^U) = \beta$?

NEYMAN CONSTRUCTION (III)

- Suppose the true value is θ_0. Depending on the observed data, one could get the intervals either indicated as **red**, or as **green** vertical lines:



Only the green confidence intervals cover the true value; red intervals are not.

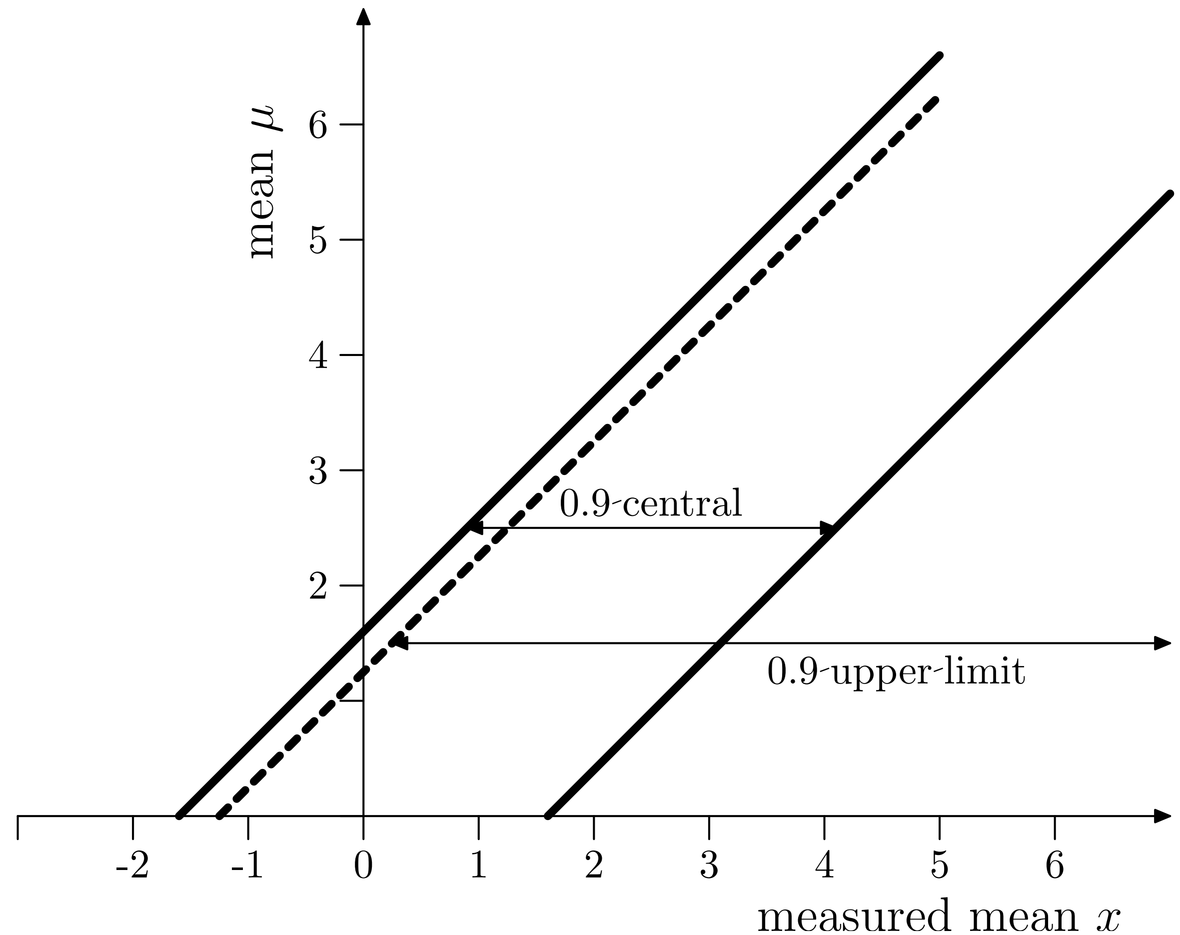
However, by definition $P(t_1(\theta)<\text{data}<t_2(\theta))=\beta$

The chance of getting a green interval is just β.

i.e. for any value of θ_0, $P(\theta_L<\theta_0<\theta^U)=\beta$

CHOICES OF INTERVAL

- As mentioned earlier, there are multiple choices of interval, even for a fixed probability β. The most common choice is the central interval, but it is also very common to discuss "one-side" only: **upper limit**.
- In particular if the parameter has a physical boundary (*e.g. cannot be negative or so*).
- Figure shows the confidence belts for a Gaussian measurement with unknown μ, and known $\sigma = 1$.



the central confidence belt
$P(\theta_L > \theta_0) = P(\theta_0 > \theta^U) = (1-\beta)/2$

versus the upper limit
$P(\theta_L < \theta_0) = \beta$ or $P(\theta_0 < \theta^U) = \beta$

CONFIDENCE INTERVALS FOR DISCRETE DATA

- Reminder: Frequentist description of the confidence interval with a coverage probability β:

$$\int_{t_1}^{t_2} f(t|\theta)dt = \beta$$

- For the case discrete observable, one has to sum over the cases to fulfill the coverage:

$$\sum_{i=L}^{U} P(t_i|\theta) \geq \beta$$

  - A slightly over-coverage may happen when adding discrete probabilities.

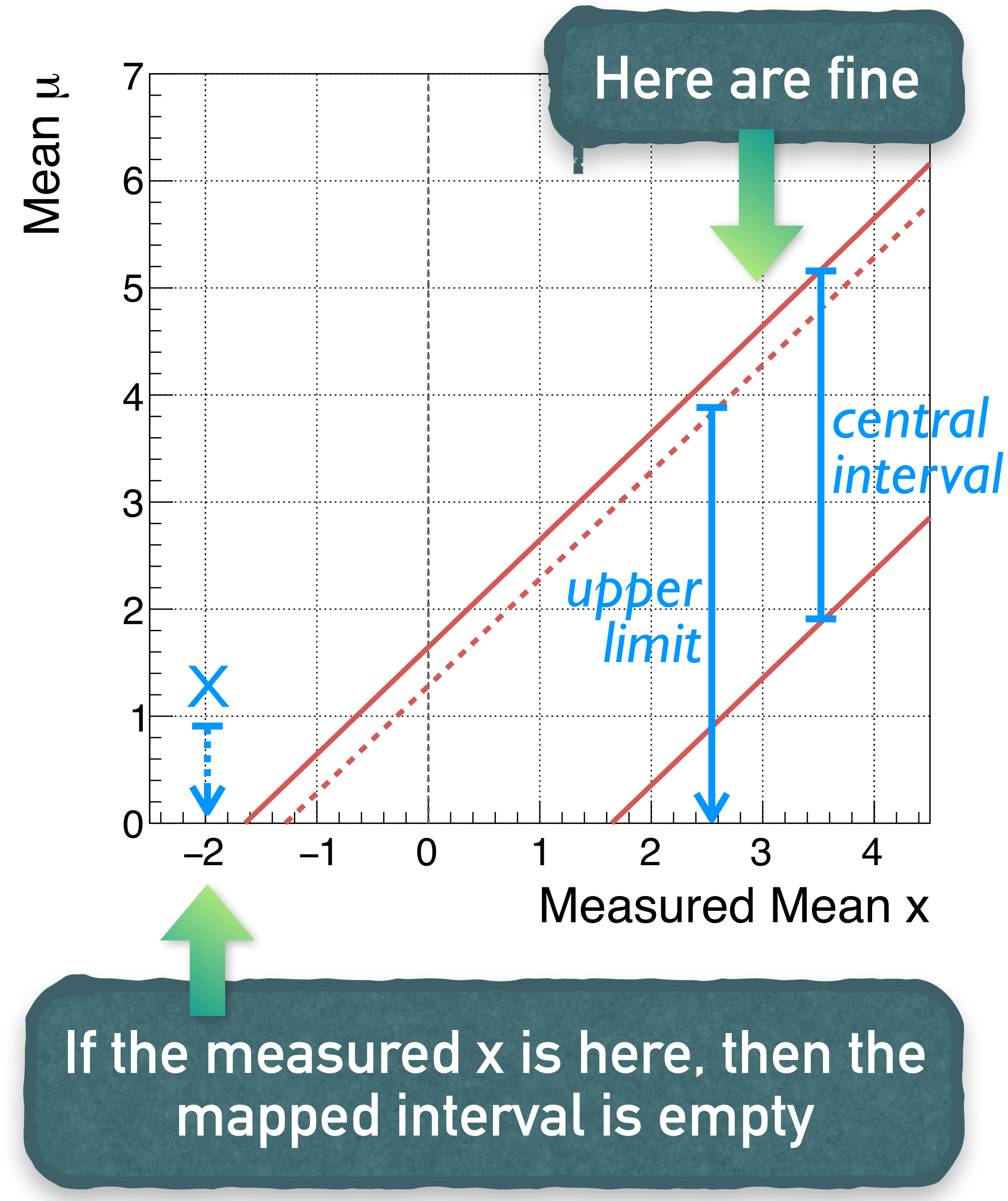
- Note the **over-coverage** case ($P>\beta$), it just loses the statistic power and results more conservative intervals, while the **under-coverage** case ($P<\beta$) is something bad and would like to be avoided definitely.

PROBLEMS IN NEYMAN CONSTRUCTION

- Consider an example of Gaussian distribution (with $\sigma=1$) with a physical boundary of $\mu>0$:

$$P(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x-\mu)^2\right]$$

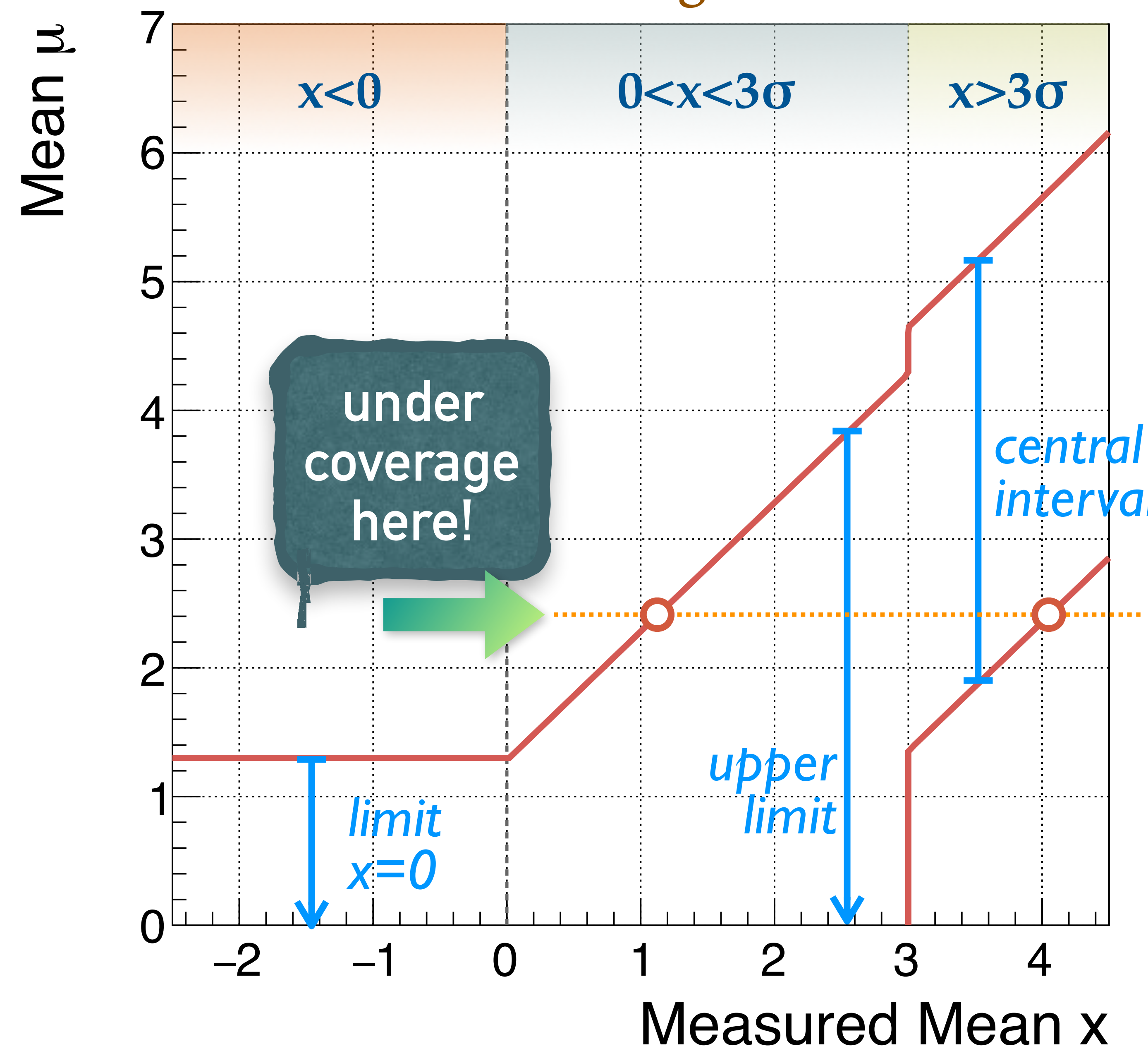
- **Empty interval**: if the measured value of x is very negative (*e.g.* $x=-2$, *due to some possible statistical fluctuations*), the resulting confidence interval is an empty set since negative values of μ are unphysical.



PROBLEMS IN NEYMAN CONSTRUCTION (II)

- **Flip-flop**: physicist tend to decide what to report based on the observed data, e.g.
  - if measured $x > 3\sigma$, report the central interval
  - if measured $x < 3\sigma$, report the upper limit
  - if measured $x < 0$ (*unphysical*), report the limit obtained from $x = 0$

flip-flop choice of interval may lead to under-coverage!

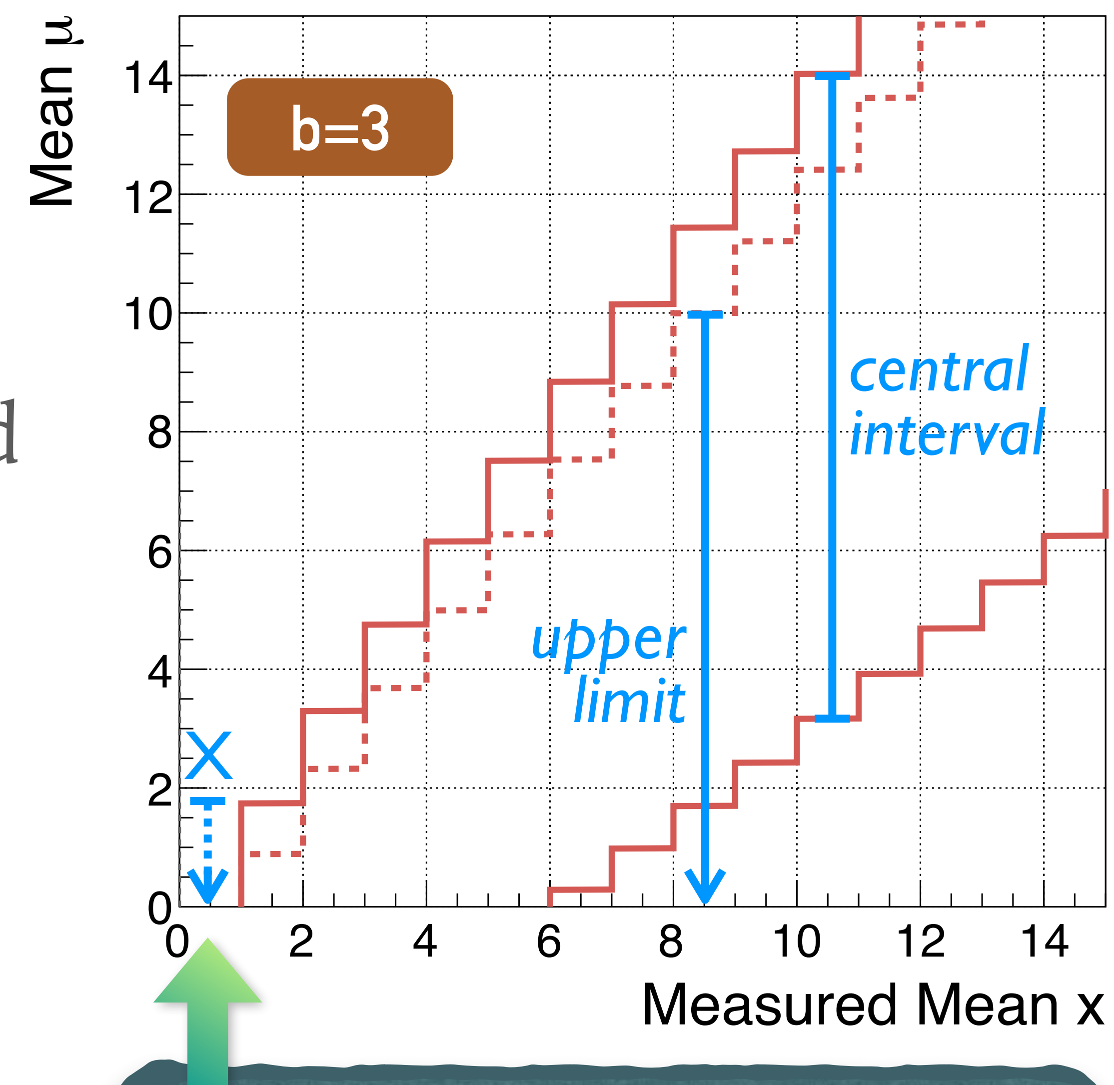


PROBLEMS IN NEYMAN CONSTRUCTION (III)

- Let's examine another example, Poisson with background ***b*=3**:

$$P(n|\mu) = \frac{(\mu+3)^n}{n!} e^{-(\mu+3)}$$

- **Empty interval**: if the measured ***n*=0**, the resulting confidence interval is an empty set since negative values of ***μ*** are unphysical.
- **Flip-flop**: choice of interval based on the observed data; lead to under-coverage.



If the measured n is 0, then the resulting interval is empty

SOLUTION: THE FELDMAN-COUSINS UNIFIED APPROACH

- Feldman and Cousins have proposed an unified approach to solve the problems (*flip-flopping and empty intervals due to physical boundary*). See **PRD 57 (1998) 3873**.
- This elegant method is to find an ordering principle which automatically gives the intervals with desired properties.
- The key idea is the **likelihood ratio ordering** principle:

$$R = \frac{f(x|\theta)}{f(x|\hat{\theta})}$$

where $f(x|\theta)$ is the likelihood of parameter θ given data x,
$f(x|\hat{\theta})$ is the maximized likelihood (over $\hat{\theta}$) for given data x.

- Then replace the interval integration by

$$\int_{t_1}^{t_2} f(t|\theta)dt \quad \Rightarrow \quad \int_{R>R_{\min}(\beta)} f(t|\theta)dt = \beta$$

integrate over the space w/ larger likelihood ratio R

FELDMAN-COUSINS FOR POISSON WITH BACKGROUND

- Let's practice the likelihood ratio ordering: Poisson with background $b=3$.
- Start with a target μ, and for each possible n, determine the $P(n|\mu)$.
- Evaluate $\hat{\mu}$ which maximize $P(n|\mu)$ and the **likelihood ratio R** is given by $R = P(n|\mu)/P(n|\hat{\mu})$.
- Add up $P(n|\mu)$ according to the R-ranking, until $\Sigma P(n|\mu) \geq \beta$
- Repeat this for all μ, then extract the interval $[\mu_L, \mu^U]$ with the same procedure.

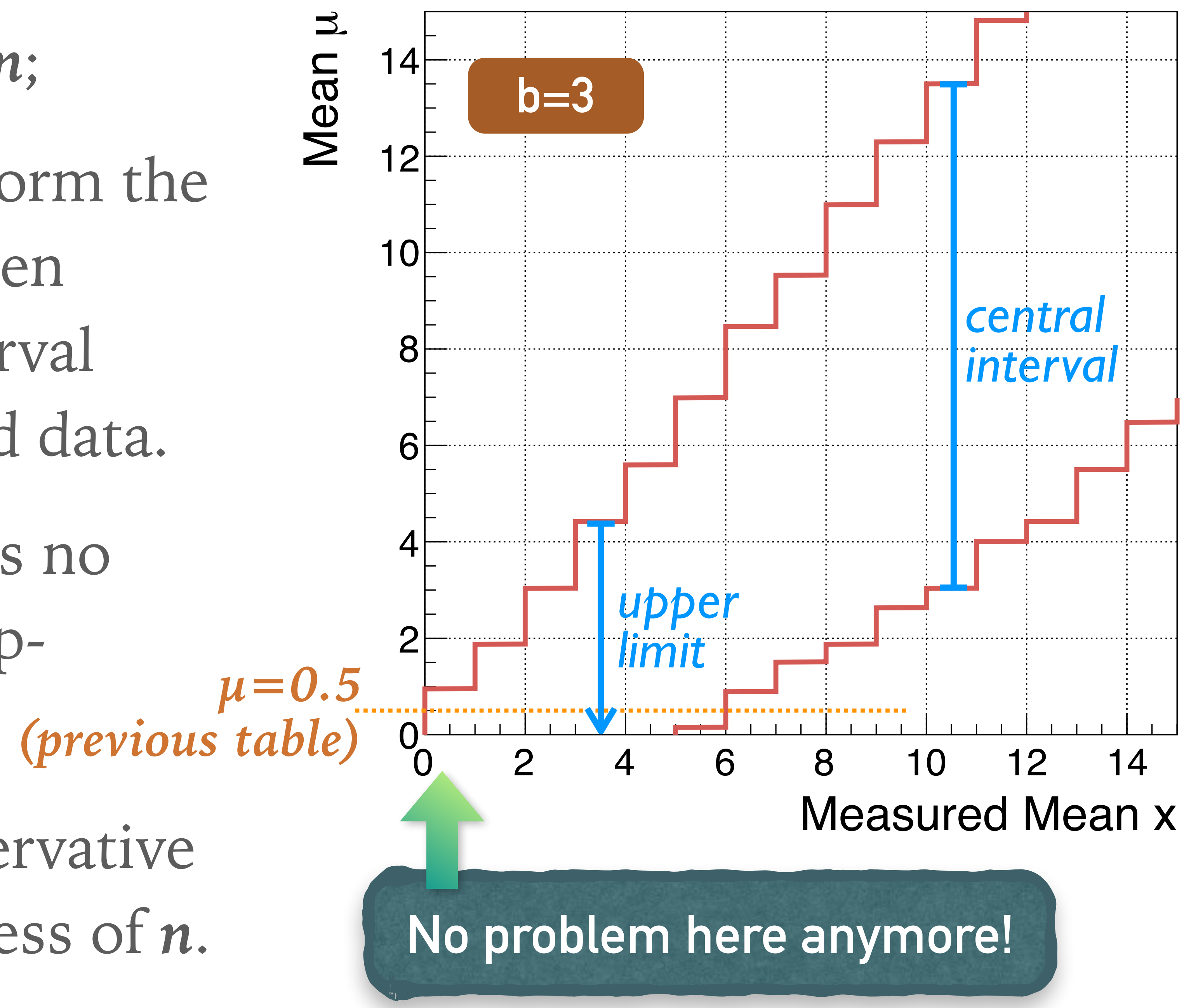
$$P(n|\mu) = \frac{(\mu+3)^n}{n!} e^{-(\mu+3)}$$

($\mu=0.5$)

n	P(n|μ)	μ̂	P(n|μ̂)	R	Rank
0	0.030	0	0.050	0.607	6
1	0.106	0	0.149	0.708	5
2	0.185	0	0.224	0.826	3
3	0.216	0	0.224	0.963	2
4	0.189	1	0.195	0.966	1
5	0.132	2	0.175	0.753	4
6	0.077	3	0.161	0.480	7
7	0.039	4	0.149	0.259	
8	0.017	5	0.140	0.121	
9	0.007	6	0.132	0.050	
10	0.002	7	0.125	0.018	
11	0.001	8	0.119	0.006	

FELDMAN-COUSINS FOR POISSON WITH BACKGROUND (II)

- For each μ, apply the likelihood ordering and evaluate the probability content over n;
- Repeat for all μ, and perform the Neyman construction, then extract the resulting interval according to the observed data.
- With this method there is no empty interval and no flip-flopping.
- Slightly falling into conservative side due to the discreteness of n.

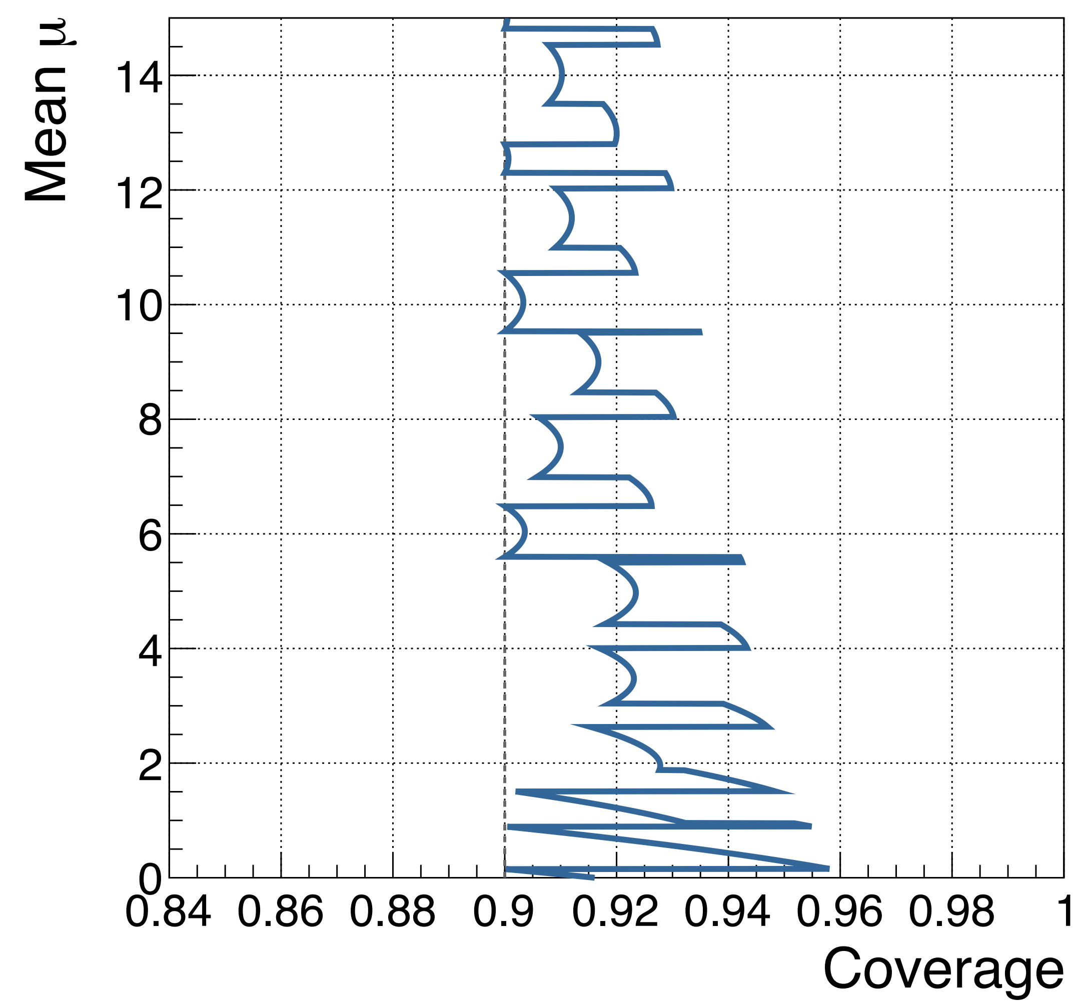


FELDMAN–COUSINS FOR POISSON WITH BACKGROUND (III)

- The Feldman-Cousins unified approach has had its greatest success when applied to Poisson data, where all previously used methods had some undesirable properties.
- The Feldman-Cousins coverage versus true mean μ "Dinosaur Plot": it shows a good coverage (all above 90% in this Poisson example with background $b = 3$). The "teeth" are just due to the discreteness of n.

F&C 90% Poisson upper bound

Observed	0	1	2	3
bkg = 0.0	2.44			
0.5	1.94	3.86		
1.0	1.61	3.36	4.91	
2.0	1.26	2.53	3.91	5.42
3.0	1.08	1.88	3.04	4.42



EXAMPLE: F&C INTERVAL FOR POISSON WITH BACKGROUND

➤ Here are an example code for the Feldman-Cousins calculation for the case of Poisson distribution with known background, using the `RooStats::FeldmanCousins` tool here!

partial example_05.cc

```
using namespace RooFit;
using namespace RooStats;

RooRealVar n("n","observed yield",0.,50.);

RooRealVar s("s","# of signal",1.,0.,10.);
RooConstVar b("b","# of background",3.);
RooAddition mean("mean","s+b",RooArgList(s,b));
RooPoisson pois("pois","Poisson PDF",n, mean);

RooDataSet data("data","data",RooArgSet(n));
n.setVal(3);
data.add(RooArgSet(n));

RooWorkspace *wspace = new RooWorkspace("wspace");
ModelConfig cfg(wspace);
cfg.SetPdf(pois);
cfg.SetParametersOfInterest(s);
cfg.SetObservables(n);
```

↖ set a constant background of 3 events

↖ also observed 3 events!

EXAMPLE: F&C INTERVAL FOR POISSON WITH BACKGROUND (CONT.)

partial example_05.cc

```
FeldmanCousins *fc = new FeldmanCousins(data,cfg);
fc->SetConfidenceLevel(0.9);
fc->UseAdaptiveSampling(true);
fc->FluctuateNumDataEntries(false);
fc->SetNBins(100); ↶ scan over true "s" with 100 bins

PointSetInterval* interval = fc->GetInterval();
printf("F&C interval: [%.2f, %.2f]\n",
     interval->LowerLimit(s),interval->UpperLimit(s));
```

```
NeymanConstruction: Prog: 100/100 total MC = 40 this test stat =
5.56255 s=9.95 [-1e+30, 1.6437]  in interval = 0

[#1] INFO:Eval -- 44 points in interval
F&C interval: [0.05, 4.55]
```

- You may find the code gives you somewhat not exactly the same results as we saw in the earlier slides!
- This is due to the fact that the `RooStats::FeldmanCousins` tool is using **Monte Carlo integration**.

BACK TO THE GAUSSIAN WITH BOUNDARY EXAMPLE

- Let's roll back a little bit to the previous example, Gaussian distribution with physical boundary at zero, and adopt the Feldman-Cousins method on it:

$$P(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(x-\mu)^2\right]$$

- For **each target μ**, find $\hat{\mu}$ which maximize $P(x|\mu)$:

$$\hat{\mu} = \begin{cases} 0 & x < 0 \\ x & x \geq 0 \end{cases} \qquad P(x|\hat{\mu}) = \begin{cases} \exp(-x^2/2)/\sqrt{2\pi} & x < 0 \\ 1/\sqrt{2\pi} & x \geq 0 \end{cases}$$

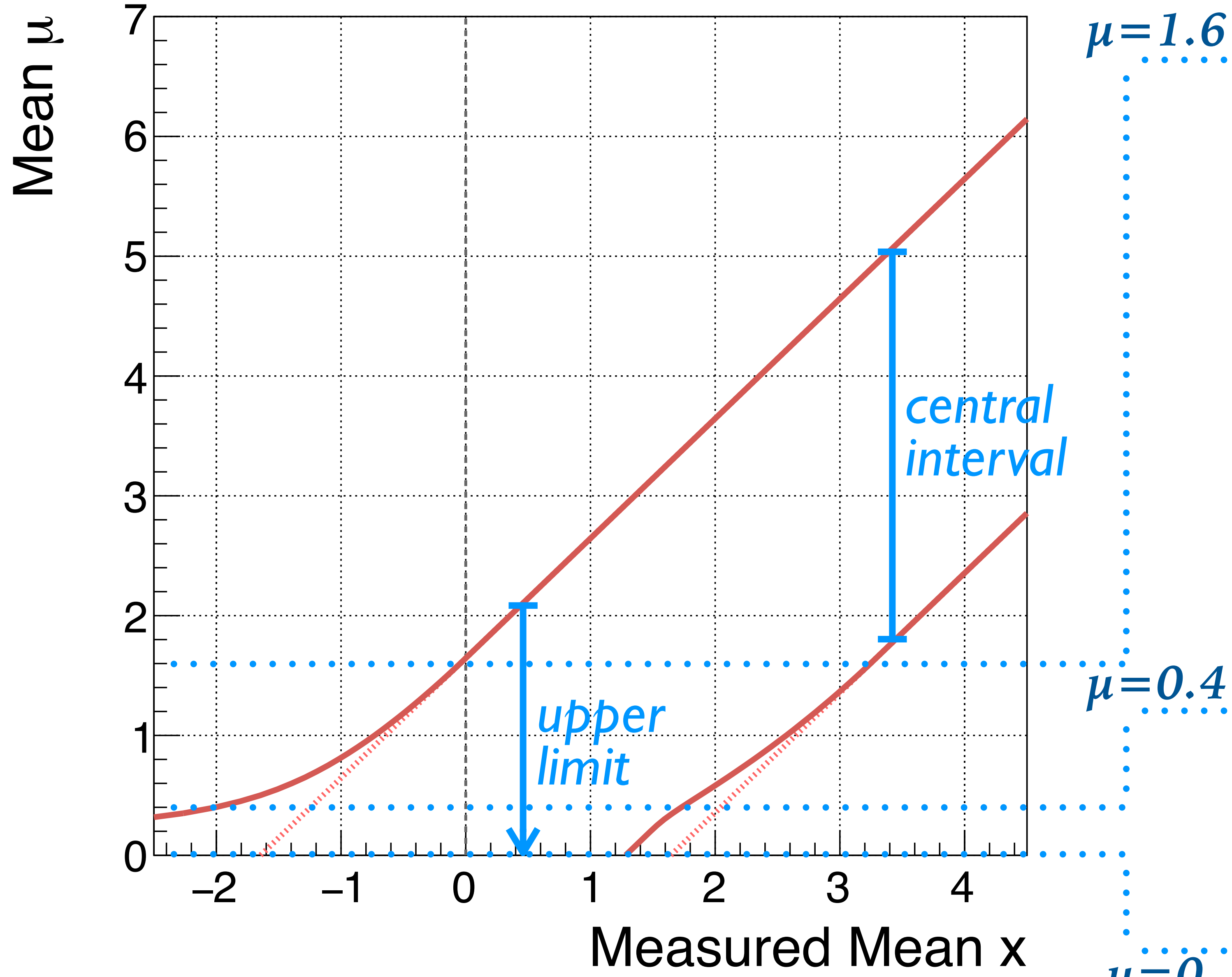
- The **likelihood ratio R** can be computed:

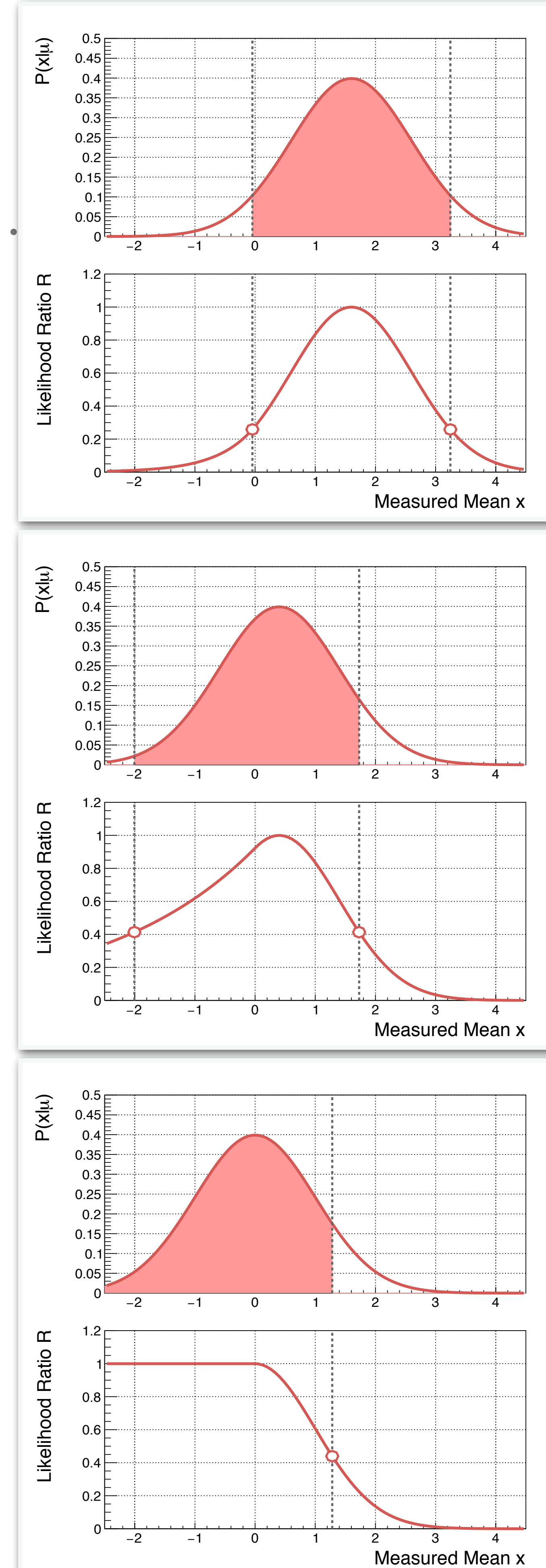
$$R = \begin{cases} \exp[-(x-\mu)^2/2]/\exp(-x^2/2) & x < 0 \\ \exp[-(x-\mu)^2/2] & x \geq 0 \end{cases}$$

- Now integrated over the R-ranked interval $[x_1, x_2]$, where

$$\int_{x_1}^{x_2} P(x|\mu)dx = \beta \quad \text{over the space } R(x) \geq R(x_1) = R(x_2)$$

GAUSSIAN WITH BOUNDARY





- **Feldman-Cousins** belts of 90% central intervals for a Gaussian measurement.
- No empty intervals and no flip-flopping.

COMMENT: NUISANCE PARAMETERS TREATMENT

- Generally in the Neyman construction, adding nuisance parameters is kind of awkward and easily gives an over-coverage result. This is due to the fact one is requiring the coverage for every possible value of the nuisance parameters (*remember our 2D Normal theory example!*).
- A proper solution has been suggested by Feldman (*which is beyond the F&C paper*), it looks for a coverage in the "worst case" of the nuisance parameter. By modify the ranking **likelihood ratio** as:

maximizes the numerator

$$R = \frac{P(x|\theta, \hat{\hat{\eta}})P(b|\hat{\hat{\eta}})}{P(x|\hat{\theta}, \hat{\eta})P(b|\hat{\eta})}$$

nuisance PDF

maximizes the denominator as usual

***θ**: signal parameter*

***η**: background (nuisance) parameter*

***x**: a measurement of θ+η*

***b**: a measurement of η*

Then proceed to the usual confidence belt construction.

In fact this is nothing different from making the nuisance parameters "profiled"!

COMMENT: UNIFIED APPROACH WITH 2 PARAMETERS

- The Feldman-Cousins approach does give good statistical properties: tight limit and correct coverage even with 2 parameters. It has been demonstrated already in the original paper.
- However, it is clear that the calculation becomes rather complicated and very difficult to use (*also requires a lot of CPU power*).
- A usual alternative solution to this is to take **profile likelihood** again, i.e. when dealing with the parameter ***X***, and make another parameter ***Y*** profiled.
- Remark: Bayesian method can be also very problematic with 2D or more. Constructing a proper multidimensional prior will pose a great problem, and hard to be "*uninformative*".

F&C CALCULATION FOR A MORE PRACTICAL CASE

- As we already pointed out, the computing for Feldman-Cousins approach is rather heavy, in particular if one really wants follow the principle of confidence belt construction:
  - For each value of true θ, scanning over the possible measure x.
  - Integrate the probabilities according to the likelihood ratio ordering until it reaches the desired probability β.
- Even if there is only one parameter of interests, the construction is still in 2D: **true θ and measure x**.
- Generally if the model is complicated (*and with multiple nuisance parameters*), a quick integration is almost impossible. In many cases this has to be carried out by Monte Carlo integration (*as you already seen when we are playing with the RooStats tool*).

F&C CALCULATION FOR A MORE PRACTICAL CASE (II)

- On the other hand, we do not really need to calculate for all possible value of measure x, given you may only want to study the situation for your observed data (*only one set / one measurement!*)
- Based on the likelihood ratio ordering, one can in fact calculate the **confidence level for any point of (x, θ)** with **toy Monte Carlo**.
- If we only scan over the true θ, only only for the observed data x, it would be a great reduction of the computing time; this is should produce a similar result as **profile likelihood scan** if everything is close to Normal distribution.
- We will demonstrate how to do this (*without RooStats tool*) in the next example.

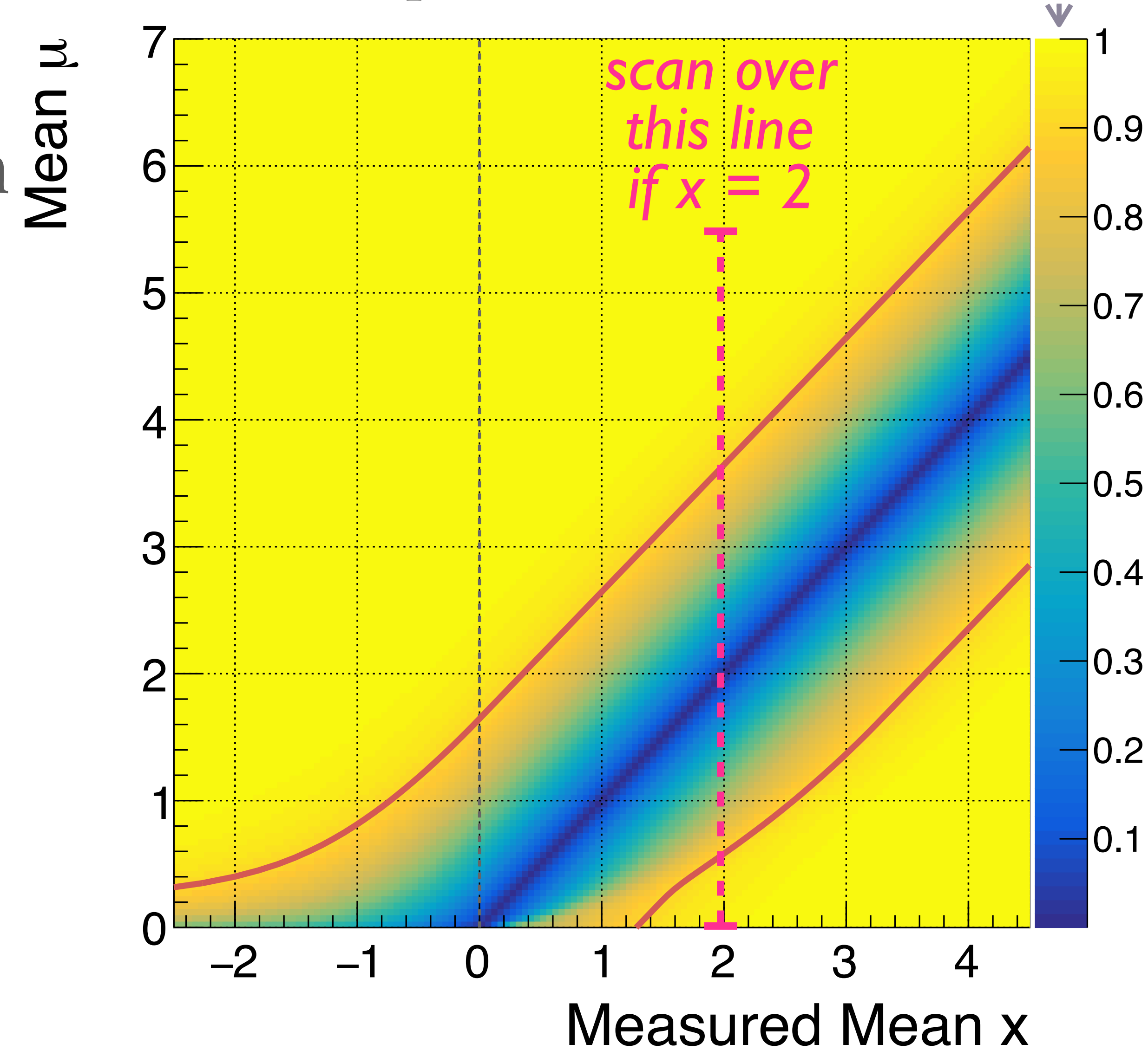
Why no RooStats tool? For easy problem this kind of heavy study is not really needed; for difficult problem it is already beyond the capability of the standard tool…

F&C CALCULATION FOR A MORE PRACTICAL CASE (III)

- Remember our confidence belt constructed for a Gaussian model with physical boundary.
- You can see the confidence belt we saw earlier is just an extraction for the points with $\beta = 0.9$.
- For a practical case, one only need to scan over the true μ for a single value of measured x, without doing the full construction of the belt.
- RooStats tool is doing the same thing in fact.

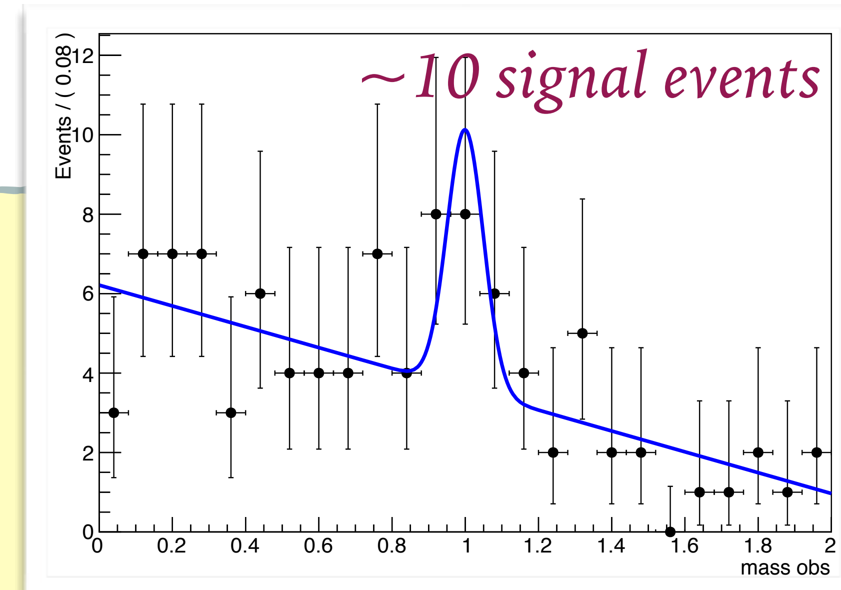
over the space $R(x) \geq R(x_1) = R(x_2)$

$$\int_{x_1}^{x_2} P(x|\mu)dx = \beta$$



EXAMPLE: STEP-BY-STEP F&C CALCULATIONS

➤ Here are a demonstration of adopting Feldman-Cousins approach. Assuming this is the data you received, and already fitted with a very simple model (note: *data file is available on the lecture web!*):



example_06.cc

```
TFile *fin = new TFile("example_06.root");
TNtupleD* nt = (TNtupleD *)fin->Get("nt");

RooRealVar mass("mass","mass obs",0.,2.);

RooRealVar mu("mu","signal mean",1.0);
RooRealVar sigma("sigma","signal width",0.05);
RooGaussian gaus("gaus","signal PDF",mass,mu,sigma);
RooRealVar slope("slope","background slope",-0.3,-5.,5.);
RooPolynomial linear("linear","background PDF",mass,RooArgSet(slope));

RooRealVar ns("ns","ns",10,0.,1000.);
RooRealVar nb("nb","nb",90,0.,1000.);
RooAddPdf model("model","PDF",RooArgList(gaus,linear),RooArgList(ns,nb));

RooDataSet data("data","data",nt,RooArgSet(mass));
model.fitTo(data,Minos(true));
```

```
  1  nb      8.97806e+01  1.00129e+01  -9.66491e+00  1.03692e+01
  2  ns      1.02353e+01  4.55361e+00  -4.21951e+00  4.90387e+00
  3  slope  -4.22156e-01  4.63662e-02  -3.87779e-02  5.51910e-02
```

- First perform a **profile likelihood scan** (*with background part being profiled!*), and convert the resulting $-2\ln L/L_{max}$ to corresponding confidence level β:

partial example_07.cc

```
void buildModel(RooWorkspace *wspace) {
    TFile *fin = new TFile("example_06.root");
    TNtupleD* nt = (TNtupleD *)fin->Get("nt");
. . . . . .
    wspace->import(model);
    wspace->import(data);

    delete fin;
}

void example_07() {
    RooWorkspace *wspace = new RooWorkspace("wspace");
    buildModel(wspace);

    RooFitResult *res0 = wspace->pdf("model")->fitTo(
        *wspace->data("data"),Save(true),Minos(true));

    TH1D *scan_2nll = new TH1D("scan_2nll","",101,-0.1,20.1);
    TH1D *scan_beta = new TH1D("scan_beta","",101,-0.1,20.1);

    for (int i=1; i<=scan_2nll->GetNbinsX(); i++) {
        wspace->var("ns")->setVal(scan_2nll->GetBinCenter(i));
        wspace->var("ns")->setConstant(true);
        RooFitResult *res1 =
             wspace->pdf("model")->fitTo(*wspace->data("data"),Save(true));
        double d2NLL = (res1->minNll()-res0->minNll())*2.;

        scan_2nll->SetBinContent(i,d2NLL);
        scan_beta->SetBinContent(i,1.-TMath::Prob(d2NLL,1));

        delete res1;
    }
```

import the model into a RooWorkspace for further use.

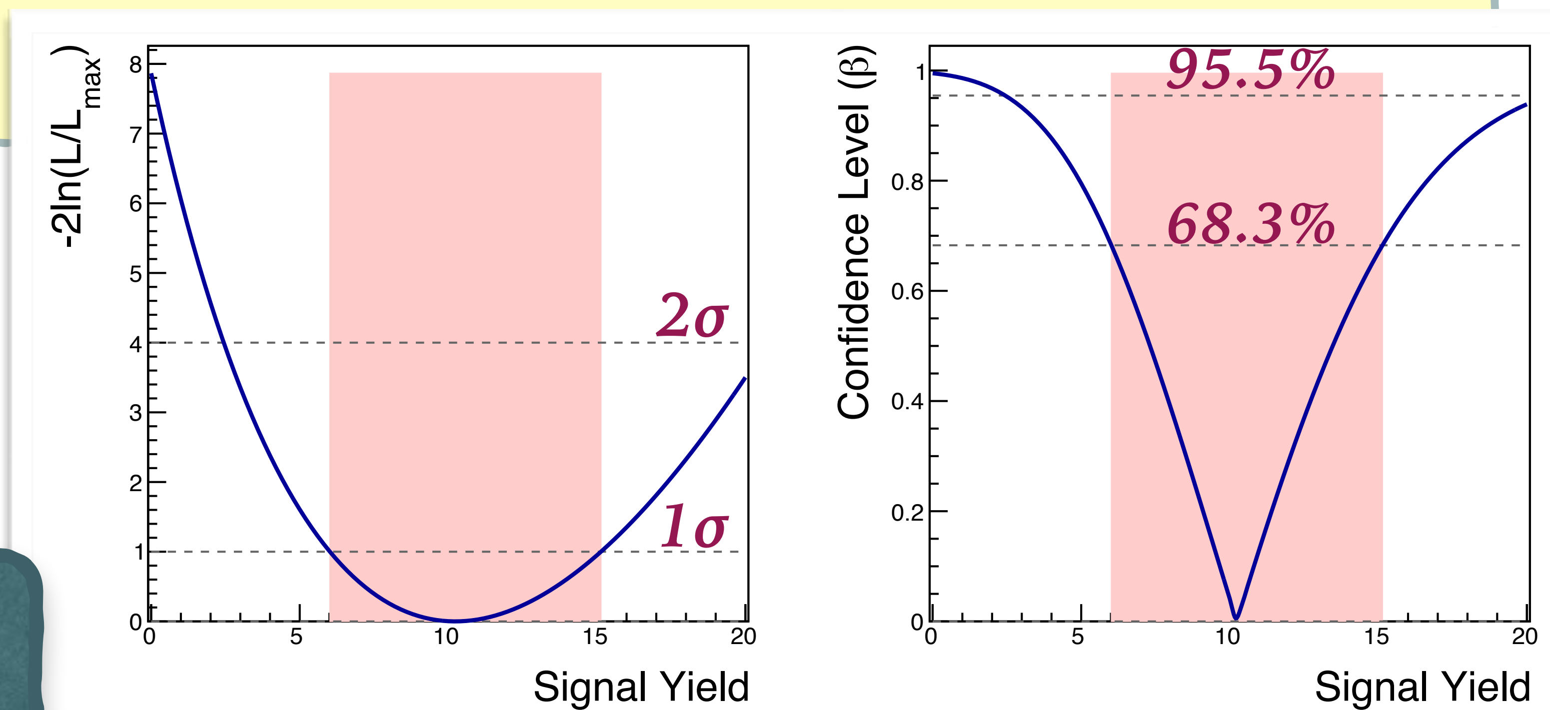
Before introducing a heavy F&C calculation, one should start from a lighter profile likelihood scan to get a rough idea!

Fill both the $-\ln(L/L_{max})$ & confidence level

```
    RooRealVar* best_ns = (RooRealVar*)res0->floatParsFinal().find("ns");

    c1->cd(1);
    scan_2nll->SetStats(false);
    scan_2nll->GetYaxis()->SetTitle("-2ln(L/L_{max})");
    scan_2nll->Draw("axis");

    box.DrawBox(best_ns->getVal()+best_ns->getErrorLo(),0.,
                best_ns->getVal()+best_ns->getErrorHi(),scan_2nll->GetMaximum());
    for(int n=0; n<=2; n++)
        lin.DrawLine(0.,n*n,20.,n*n);

    scan_2nll->Draw("csame");

    c1->cd(2);
    scan_beta->SetStats(false);
    scan_beta->GetYaxis()->SetTitle("Confidence Level (#beta)");
    scan_beta->Draw("axis");

    box.DrawBox(best_ns->getVal()+best_ns->getErrorLo(),0.,
                best_ns->getVal()+best_ns->getErrorHi(),scan_beta->GetMaximum());
    for(auto prob: {0.,0.6827,0.9545})
        lin.DrawLine(0.,prob,20.,prob);

    scan_beta->Draw("csame");
}
```

partial example_07.cc

One can find a good match between MINOS error band, and the 68% C.L. cross points.



Proceed to F&C study now!

EXAMPLE: STEP-BY-STEP F&C CALCULATIONS (CONT.)

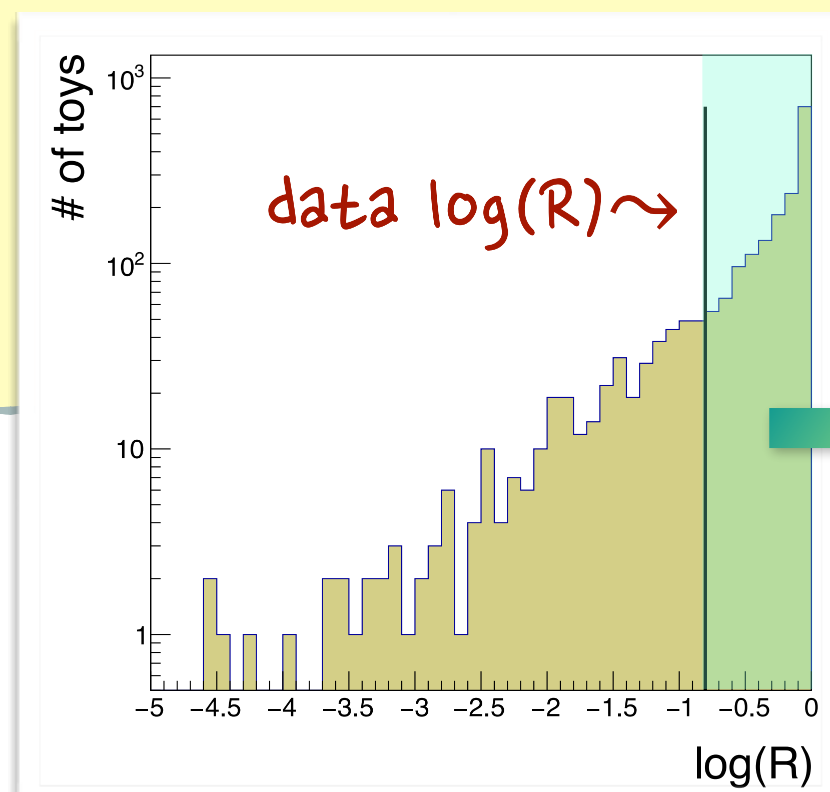
- Now let's perform a Feldman-Cousins study for a given target (*true*) signal strength.
- The procedure can be carried out as following:
  - **Fit to data with the signal fixed to the target value**, using this resulting model to generate toy data sets.
  - For each set of toy data, two fits are performed: **one with signal fixed, one with signal floated**. Calculate the difference in the resulting $-\ln(L)$ value, this actually gives the expected distribution of $\ln(R)$, where R is the **likelihood ratio for F&C ordering**.
  - Perform the same two fits to data and obtain the $\ln(R)$ for data.
  - The fraction of toy sets which has a **$\ln(R)$ value greater than the value from data** gives the estimate of confidence level β at the target signal strength and with the observed data.

Example code is given in the next page.

partial example_08.cc

```
double FC_scan(double target_ns = 10., int ntoys = 100)
{
    RooWorkspace *wspace = new RooWorkspace("wspace");
    buildModel(wspace);

    RooFitResult *res0 = wspace->pdf("model")->fitTo(*wspace->data("data"),Save(true));
    wspace->var("ns")->setVal(target_ns);
    wspace->var("ns")->setConstant(true);
    RooFitResult *res1 = wspace->pdf("model")->fitTo(*wspace->data("data"),Save(true));
    double logR_data = res0->minNll()-res1->minNll();

    double beta = 0.;
    for (int idx=0; idx<ntoys; idx++) {
        RooWorkspace* wspace_toy = new RooWorkspace(*wspace);
        RooDataSet *toy = wspace_toy->pdf("model")->generate(*wspace_toy->var("mass"));

        RooFitResult *res1 = wspace_toy->pdf("model")->fitTo(*toy,Save(true));
        wspace_toy->var("ns")->setConstant(false);
        RooFitResult *res0 = wspace_toy->pdf("model")->fitTo(*toy,Save(true));

        double logR = res0->minNll()-res1->minNll();
        if (logR>logR_data) beta += 1.;

        delete . . .
    }
    beta /= (double)ntoys;

    delete . . .
    return beta;
}
```

produce test statistics w/ toy

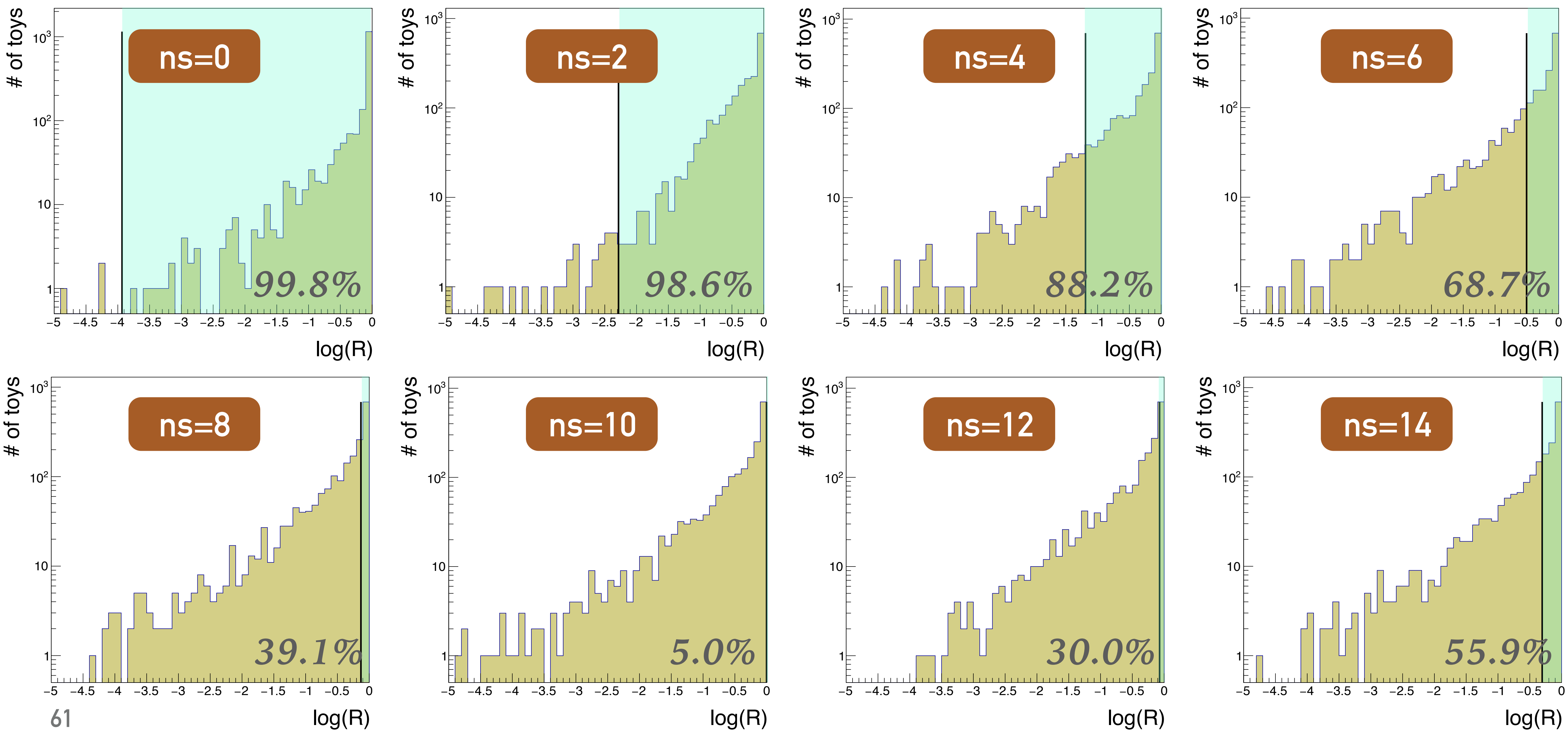


Test statistics distribution for # of target signal = 5

This region gives an **estimation of confidence level.** e.g. 79% of toys are here, thus β=0.79.

SCAN OVER TARGET (TRUE) SIGNAL

- This is what one should be able to observe: when the scanned point of signal strength is away from the data fitted value, the probability content increases.



F&C APPROACH VERSUS PROFILE LIKELIHOOD SCAN

- Now we can easily calculate the confidence level for any given target signal strength using the F&C approach. The results can be compared with the result of likelihood scan:

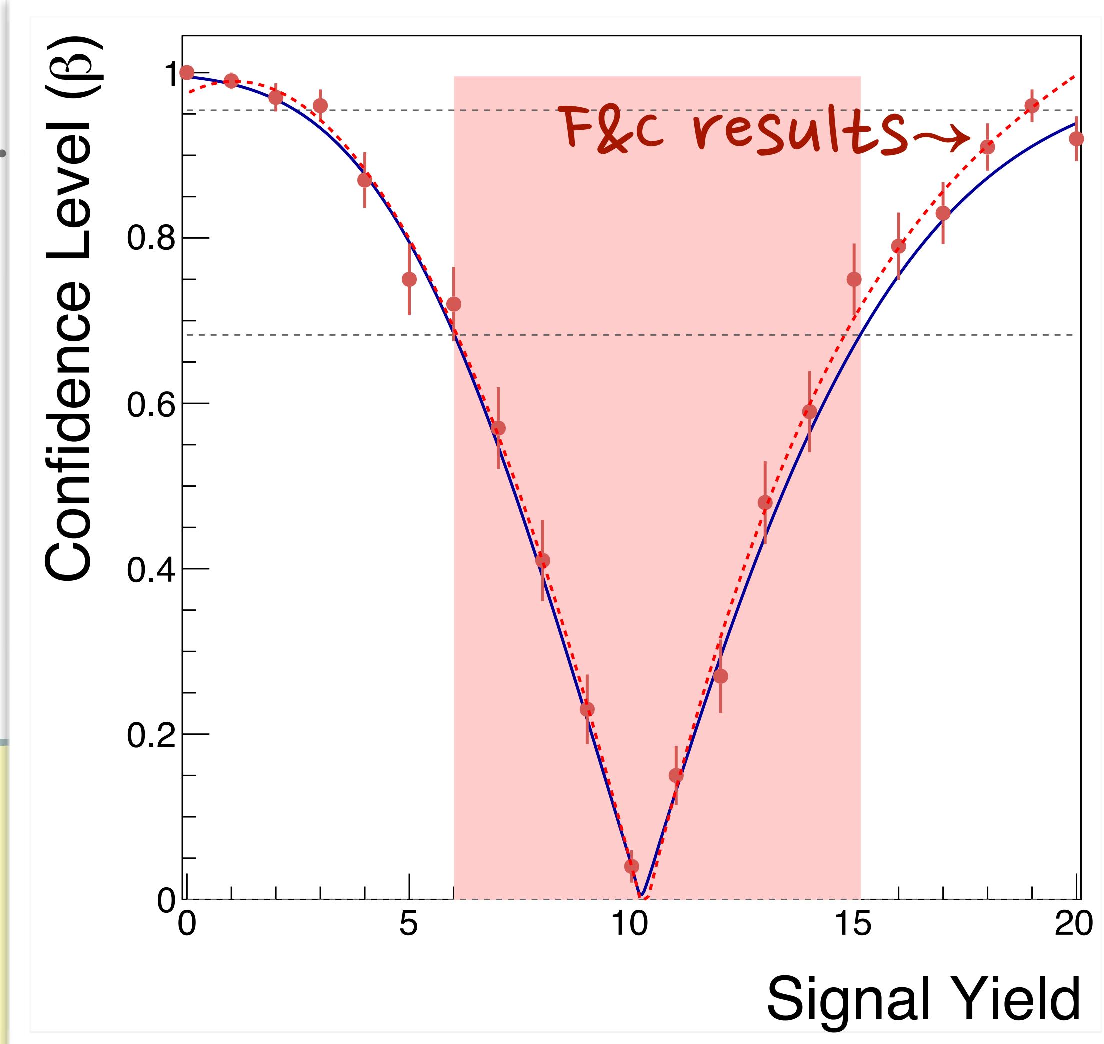
partial example_08.cc

```
void example_08()
{
    RooWorkspace *wspace = new RooWorkspace("wspace");
    buildModel(wspace);

    RooFitResult *res0 = wspace->pdf("model")->fitTo(
        *wspace->data("data"),Save(true),Minos(true));

    TH1D *scan_beta = new TH1D("scan_beta","",101,-0.1,20.1);
    TH1D *scan_fc = new TH1D("scan_fc","",101,-0.1,20.1);

    for (int i=1; i<=scan_beta->GetNbinsX(); i++) {
        wspace->var("ns")->setVal(scan_beta->GetBinCenter(i));
        wspace->var("ns")->setConstant(true);
        RooFitResult *res1 = wspace->pdf("model")->fitTo(
            *wspace->data("data"),Save(true));
        double d2NLL = (res1->minNll()-res0->minNll())*2.;
        scan_beta->SetBinContent(i,1.-TMath::Prob(d2NLL,1));

        if ((i%5)==1) {
            double beta = FC_scan(scan_beta->GetBinCenter(i),100);
            double beta_err = sqrt(beta*(1.-beta)/100);
            scan_fc->SetBinContent(i,beta);
            scan_fc->SetBinError(i,beta_err);
        }
    }
```

call the FC scan function in steps of 1 event

F&C VERSUS PROFILE LIKELIHOOD

➤ Let's do a little bit interpolation to get the confidence intervals from F&C results (*very consistent with MINOS = profile likelihood!*)



partial example_08.cc

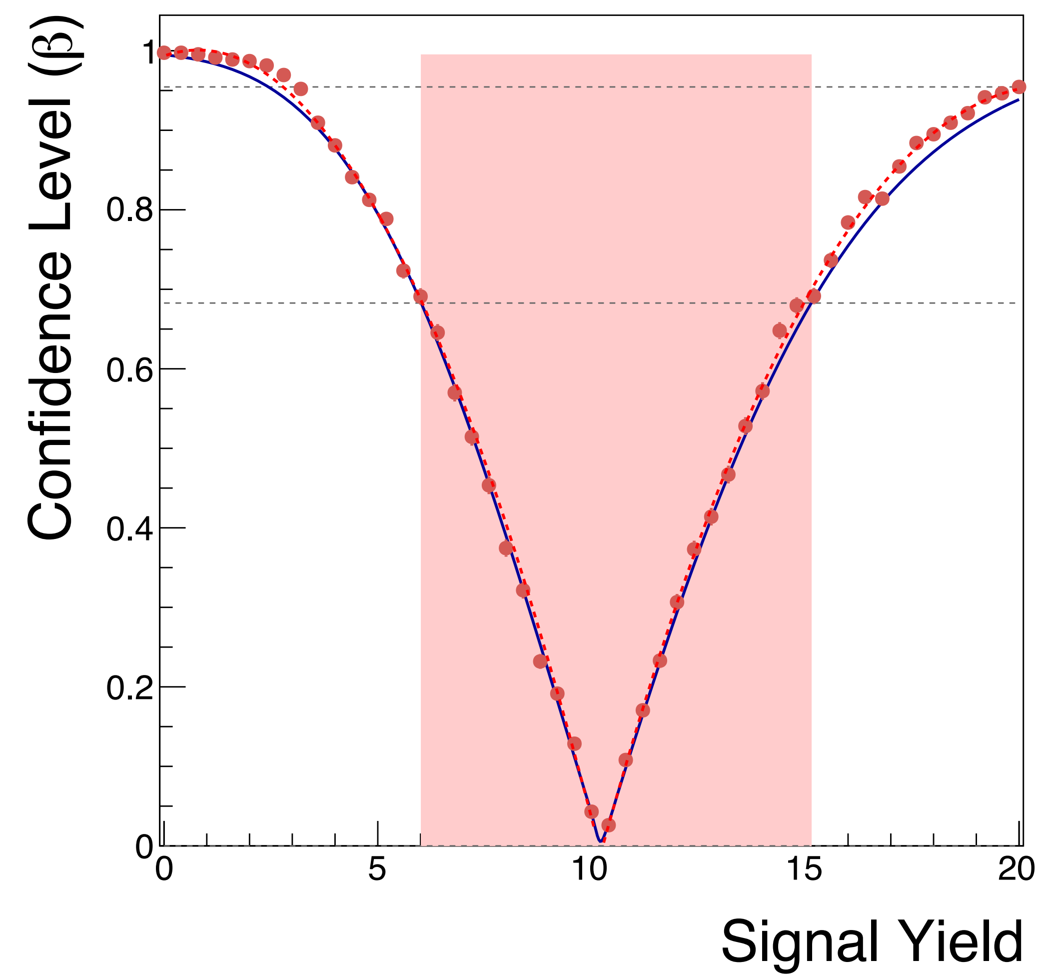
```
. . . . . .
    scan_beta->Draw("csame");
    scan_fc->SetMarkerStyle(20);
    scan_fc->Draw("esame");

    TF1 fl("fl", "pol3", 0.,best_ns->getVal());
    TF1 fr("fr", "pol3",best_ns->getVal(),20.);
    scan_fc->Fit("fl","+R");
    scan_fc->Fit("fr","+R");

    printf("MINOS 68.3%% interval: [%.2f, %.2f]\n",
           best_ns->getVal()+best_ns->getErrorLo(),
           best_ns->getVal()+best_ns->getErrorHi());
    printf("F&C   68.3%% interval: [%.2f, %.2f]\n",
           fl.GetX(0.6827),fr.GetX(0.6827));
}
```

↶ fit the scanning points w/ 3rd order polynomial

```
MINOS 68.3% interval: [6.02, 15.14]
F&C   68.3% interval: [6.07, 14.79]
```

COMMENT: STEP-BY-STEP F&C CALCULATIONS

- We have demonstrated how to perform a confidence interval calculation using the unified approach, by ourselves.
- The calculation is heavy even with such a simple model and small data set. The real application will require far more computing power.
- The resulting confidence interval is generally consistent (with only minor deviations) with the profile likelihood scan, which is *asymptotically* valid.



A finer F&C scan with more toys! Start to see some deviations from likelihood method!

COMMENT: NUISANCE PARAMETERS IN TOY MC

- In the previous example there are two floated nuisance parameters (`nb` and `slope`), in general there is no special treatment but just have them profiled during the generation of the test statistics distribution.
- However if there is any constrained parameter, **the constraint PDF term has to be randomized for each toy data as well**. For example, consider a Gaussian constrained likelihood:

  $$L' = L(X|\theta, \lambda) \times P(\lambda|\mu_\lambda, \sigma_\lambda)$$

  The model $\boldsymbol{P(\lambda)}$ is generally not handled by RooFit in the event generation. Thus one has to either randomize the constrained mean (μ_λ) for the fitting model or λ value in the generation for each set of toy, according to the constraint PDF.
- Otherwise the toy data will not have the proper statistical behavior!

COMMENT: ISSUES OF THE UNIFIED APPROACH

- As we already see: the Feldman-Cousins unified approach solves the main problems when the parameter close to its physical boundary, within the framework of the classical Neyman construction. It also ensures a proper statistical coverage!
- **However there are some issues still:**
  - constructing the confidence intervals is rather complicated, CPU-intensive calculation are generally required (*e.g. large toy Monte Carlo set generation*);
  - In case of zero observed events, gives better limits for experiments that expect higher background unlike Bayesian method with uniform prior (*although this also happens for the case of non-uniform prior*).

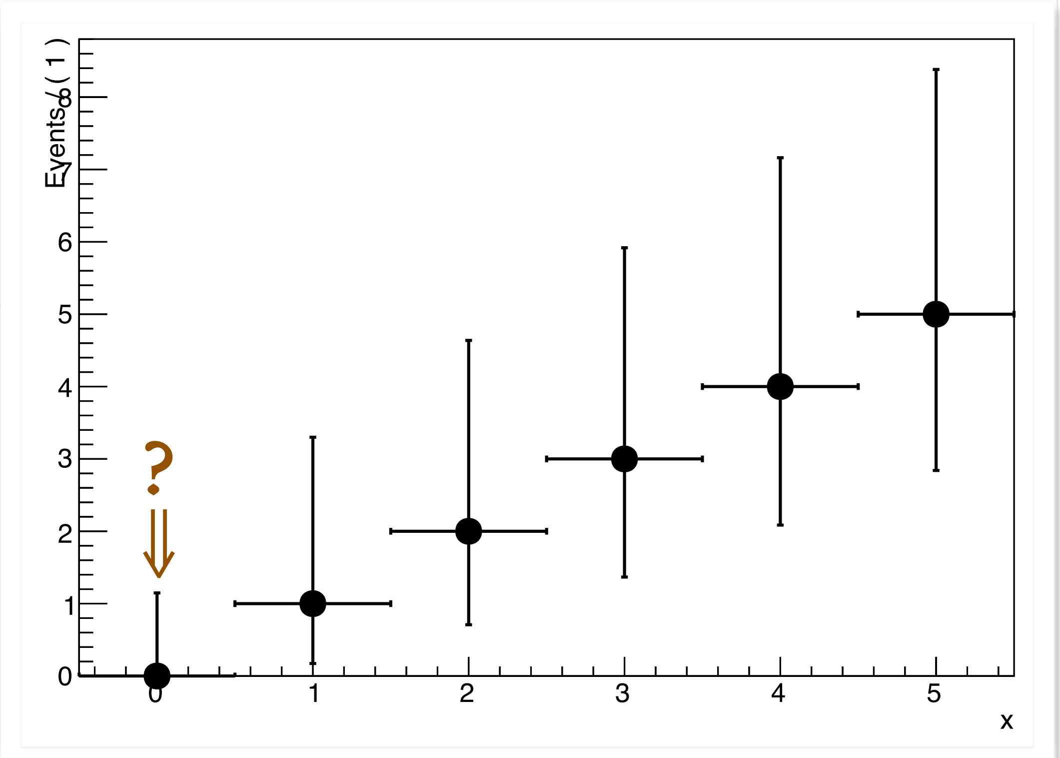
COMMENT: PROBLEMS WITH FREQUENTIST METHODS

- Just using the pure unified approach to estimate the upper limits may give problematic results in the presence of background.
  - In some cases, people found a statistical (*under-*)fluctuation of the background may lead to the exclusion of zero signal, which is unphysical.
    - ➡ Information is not sufficient to discriminate the **b-only** and **s+b** hypotheses.
  - Also, in some of the cases, when adding the channels with low signal sensitivity, may produce upper limits that are worse than without adding them.
- This is the reason why people introduce a modified frequentist method at LEP and LHC: **the CLs method**. Which will be discussed in the next lecture.

Not yet at the end of story!
We will continue to discuss hypothesis testing &
upper limit calculation again in the next lecture!

COMMENT: ROOFIT ASYMMETRIC ERROR BARS?

- You may have notice one thing: RooFit actually put asymmetric errors on data points by default.
- Now we can finally come back to discuss this. **Why/how RooFit obtain those asymmetric errors and put on the figures?**
- An example code to demo:



example_09.cc

```
TH1D *hist = new TH1D(“hist",
     "test histogram",6,-0.5,5.5);
for(int i=0;i<6;i++)
     hist->SetBinContent(i+1,i);

RooRealVar x("x","x",-0.5,5.5);
RooDataHist data("data","data",x,hist);

RooPlot *frame = x.frame();
data.plotOn(frame,LineWidth(2),MarkerSize(2.));
frame->Draw();
```

*error bar for **n=0** is very tricky! Will discuss it later.*

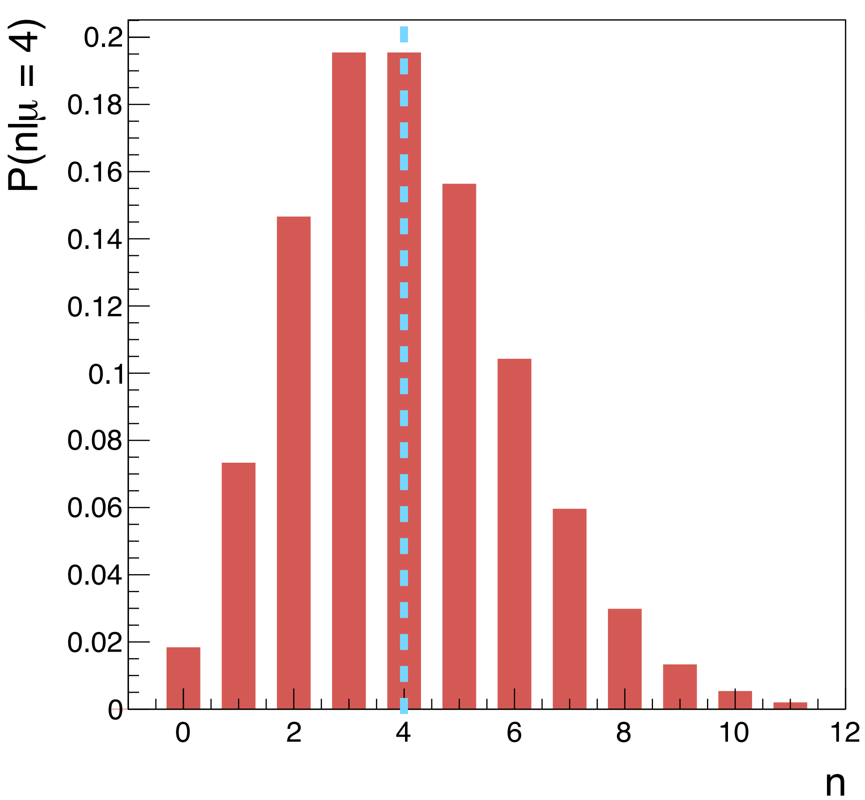
(A)SYMMETRIC ERROR BARS

- We have to come back to the Poisson distribution:

$$p(n|\mu) = \frac{\mu^n e^{-\mu}}{n!}$$

This gives the probability of expected mean value μ and finding n.

- The usual **square-root-of-n** error is obtained by just setting μ to be observed value n, and take the variance of the distribution. For example, you find $n=4$, and set $\mu=n=4$:



Given variance is 4, so the "error bar" is ±2. But Poisson distribution is asymmetric…

$p(4|\mu=2) = 0.090$

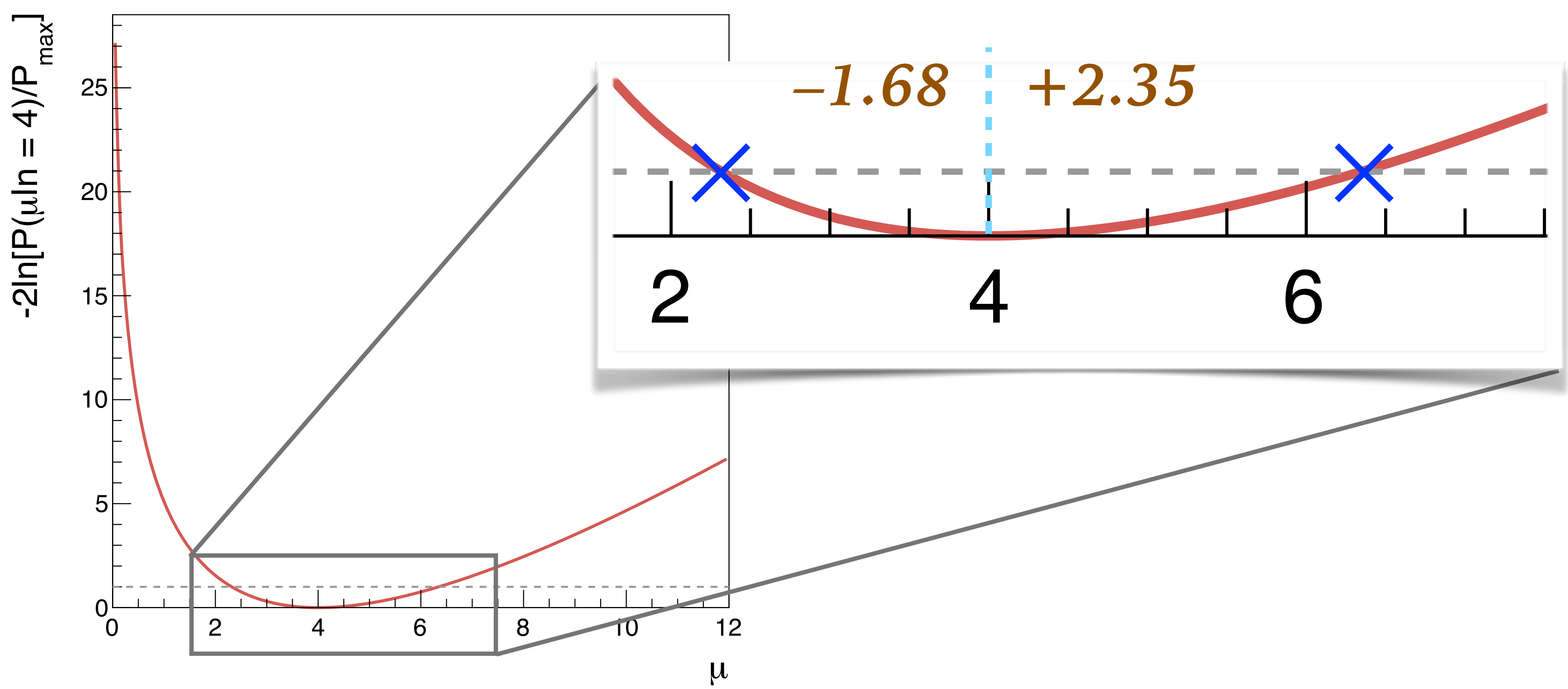
$p(4|\mu=6) = 0.134$

ALTERNATIVE OPTION

- Based on what we discussed earlier in this lecture, inserting the observed ***n*** into the Poisson PDF:

$$L(\mu) = \frac{\mu^n e^{-\mu}}{n!}$$

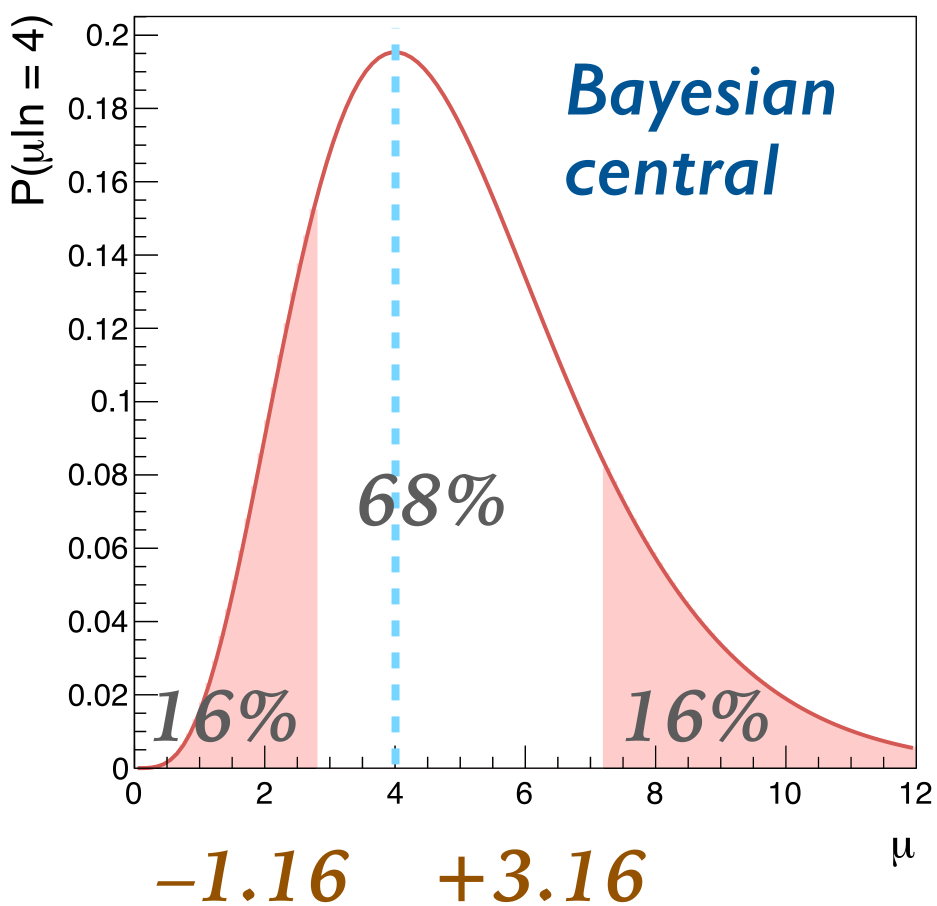
- Basically this is a **likelihood function of *μ***. So let's do a likelihood scan as we introduced earlier in this lecture?

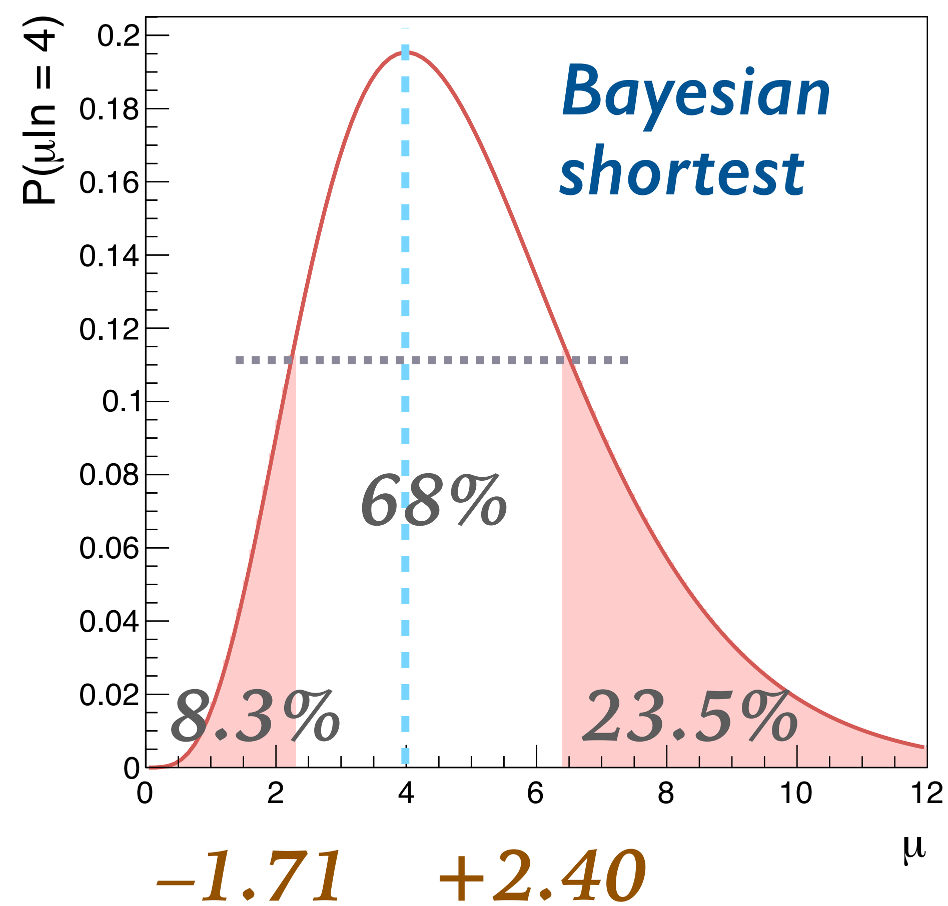


MORE ALTERNATIVE OPTIONS?

- Well, we have discussed the Bayesian method, let's take the **posterior probability** with an uniform prior, one can at least come up with two more options:

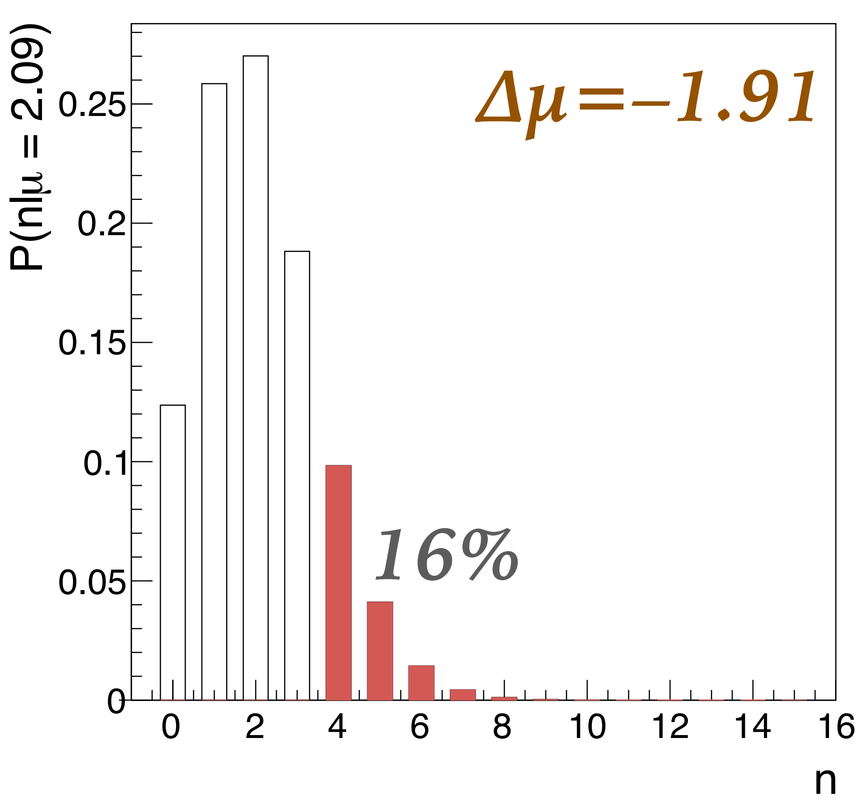
$$p(\mu|n) = \frac{\mu^n e^{-\mu}}{n!}$$

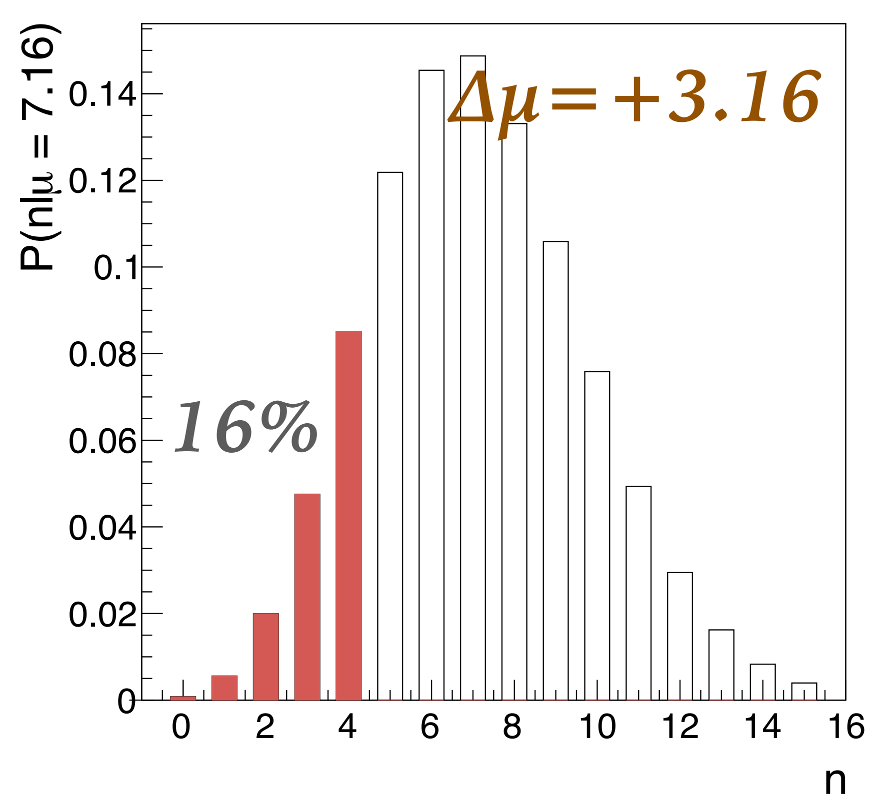




THE FREQUENTIST APPROACH

- One can also adopt the full Neyman construction here, but since we only need to calculate the interval for a fixed given observed ***n***, this can be done easily — find extreme values of μ that are still being compatible with observed yield:
  - As $\mu > 7.16$, the probability to observe 4 events (or less) is $< 16\%$;
  - As $\mu < 2.09$, the probability to observe 4 events (or more) is $< 16\%$.

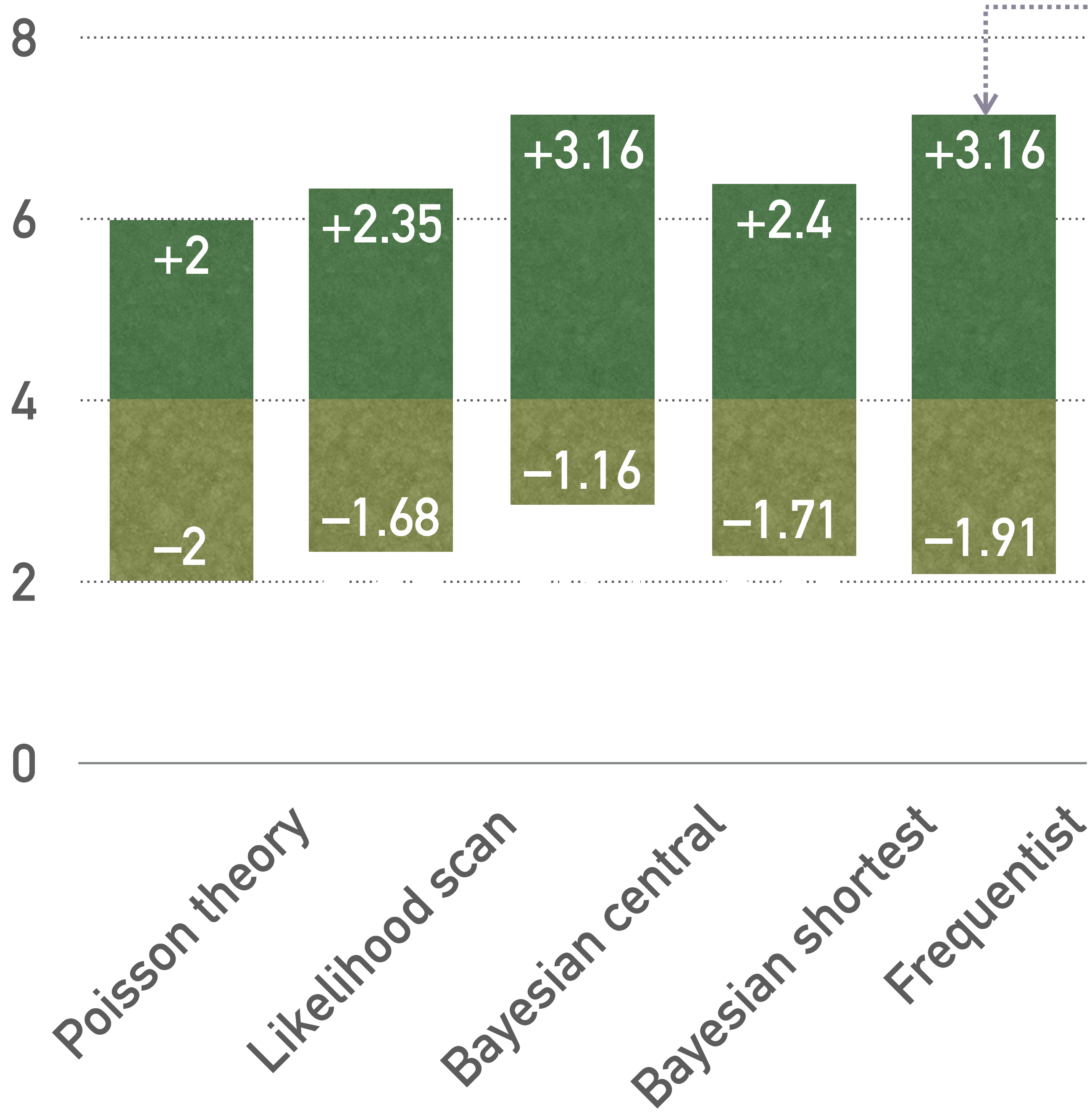


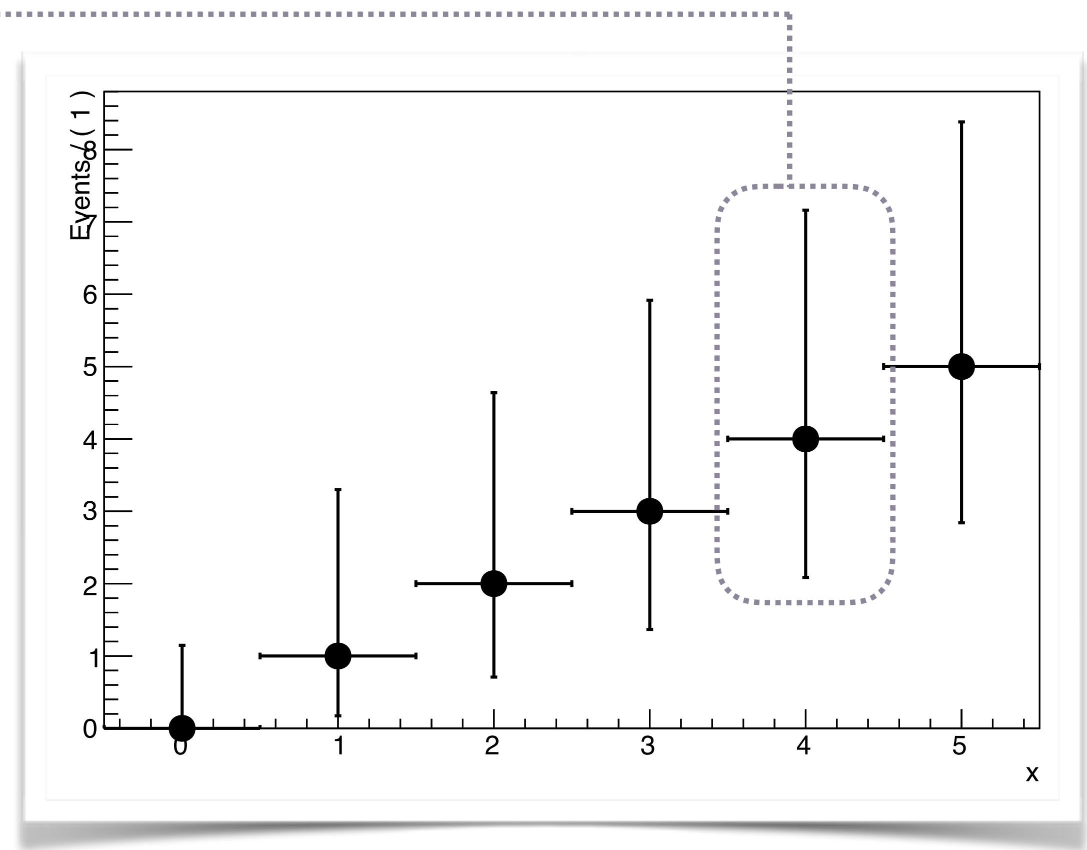


$$p(n|\mu) = \frac{\mu^n e^{-\mu}}{n!}$$

ROOFIT CHOICE

➤ Let’s summarize all these different intervals:

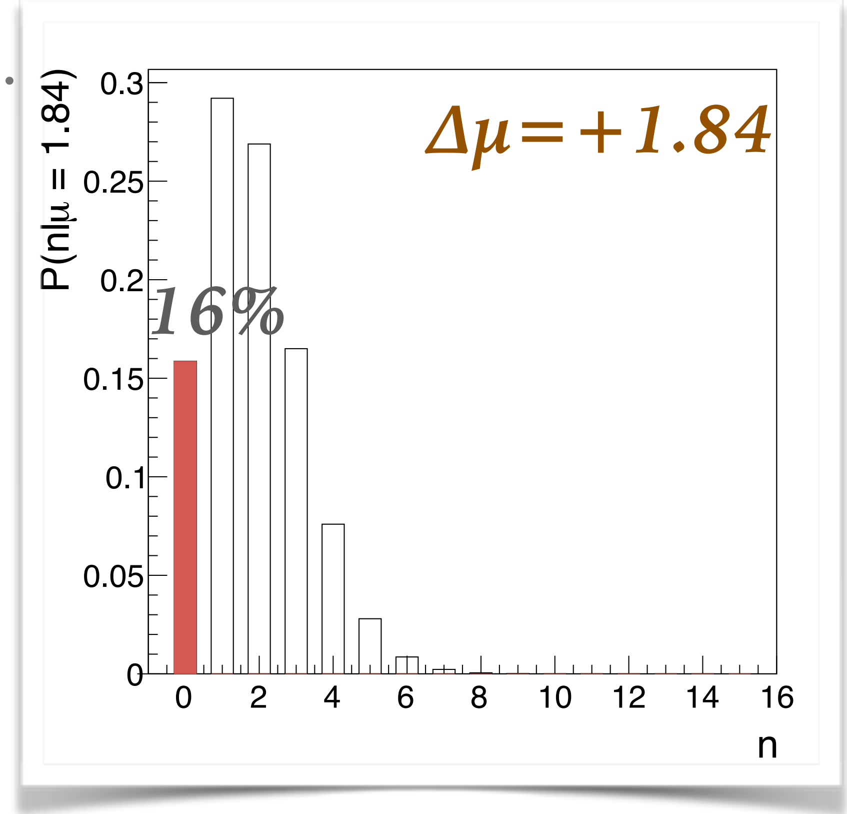




RooFit chooses the **frequentist approach** as the default.

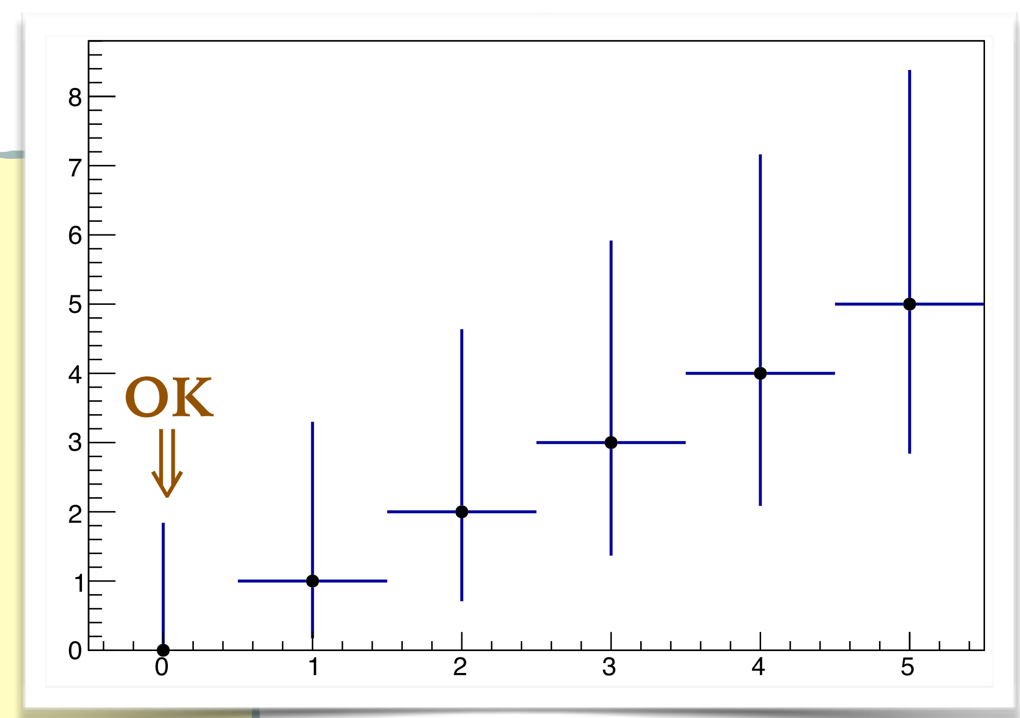
ROOFIT CHOICE (CONT.)

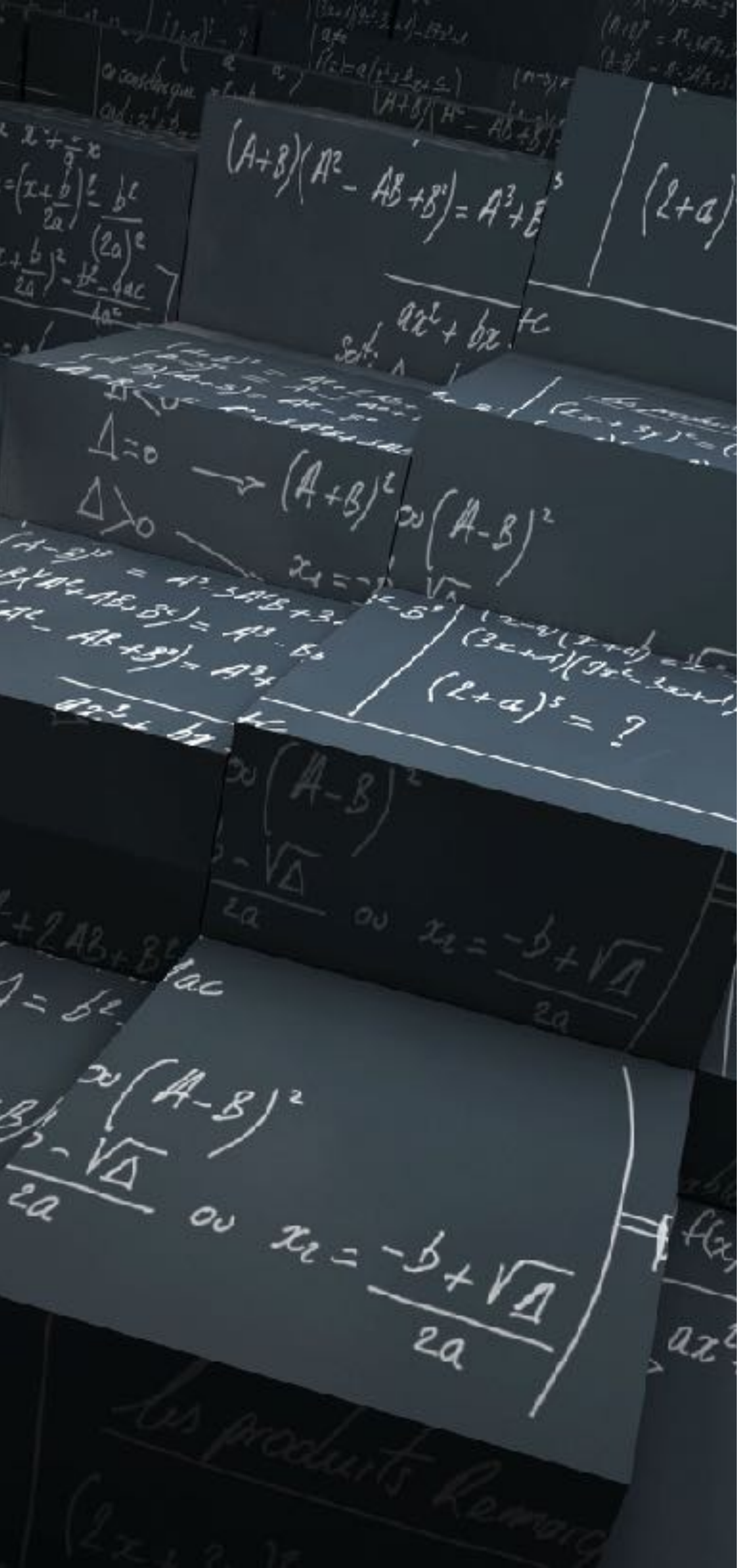
- But there is an inconsistency for the case of $n=0$. This is due to the flip-flopping between 1-side and 2-side intervals.
- RooFit draws a positive error bar of around **+1.2**, but it should be in fact around **+1.8** if we adopt the same recipe as discussed above.
- The "correct" interval has been implemented within ROOT already.



example_09a.cc

```
TH1D *hist = new TH1D("hist",
     "test histogram",6,-0.5,5.5);
for(int i=0;i<6;i++)
     hist->SetBinContent(i+1,i);

hist->SetBinErrorOption(TH1::kPoisson);
hist->SetLineWidth(2);
hist->SetMarkerStyle(20);
hist->Draw("e0");
```





SUMMARY

- In this lecture several methods of interval estimation have been discussed, including both Bayesian methods and frequentist methods.
- This should give you a more general picture behind just quoting an error bar from your fitter.
- For the next lecture, we are going to discuss the hypothesis test, and will touch the upper limit calculation again.